TSBB21, Lecture 5
Image Formation

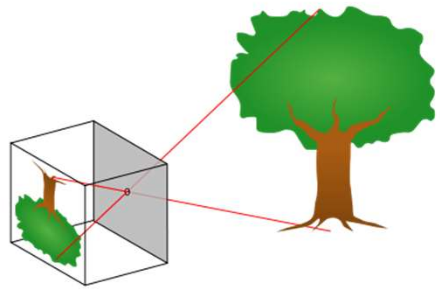
- Diffraction-limited systems
- A lens produces the Fourier transform. And more.
- The Airy disk
- Optical transfer function (OTF)
- The point spread function
  - Airy disk
  - Out-of-fucus blur
- Depth of field, Circle of confusion
- F-number
- Lens distorsion
- Vignetting and the $\cos^4$ law
- Chromatic aberration

- Literature:
  - Canon Europe: Optical Terminology
  - Cos4 Law: Derivation of the Cos4 Law
  - P. Danielsson: Optiska system
  - R. Forchheimer: Härledning av PSF för en tunn lins

- Thanks to:
  - Klas Nordberg: Initiated this course. Many slides in this lecture are similar to his slides.
  - Robert Forchheimer: Especially for showing that a lens produces the Fourier transform.

From Lecture 1: Lenses vs. infinitesimal aperture

- The pinhole camera is an ideal model of the *camera obscura*



- The pinhole camera model can also be utilized for a camera with one lens or a system of lenses provided that we consider:
  - The lens camera only gives a perfect sharp image for objects in the object plane, at distance a from the lens.
  - Then the image plane is located at the distance b from the lens. Note that since normally $a >> b$ => $b \approx f$, where f is the focal length.

The lens law:

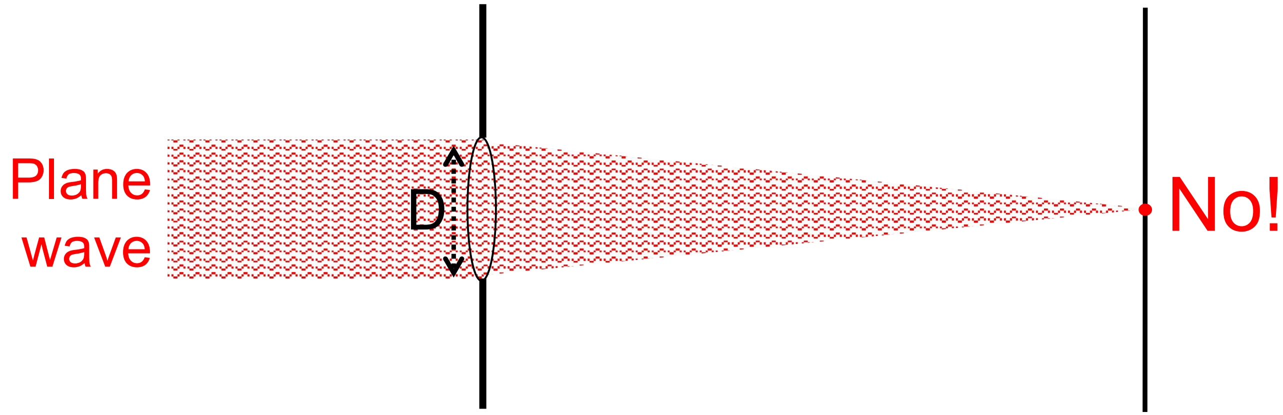
$$\frac{1}{a} + \frac{1}{b} = \frac{1}{f}$$

Diffraction limited systems

- Due to the wave nature of light, even when various lens effects are eliminated, light from a single 3D point cannot be focused to an arbitrarily small point if it has passed an aperture
- For coherent light:
  - Huygens' principle: treat the incoming light as a set of point light sources
  - Gives *diffraction* pattern at the image plane

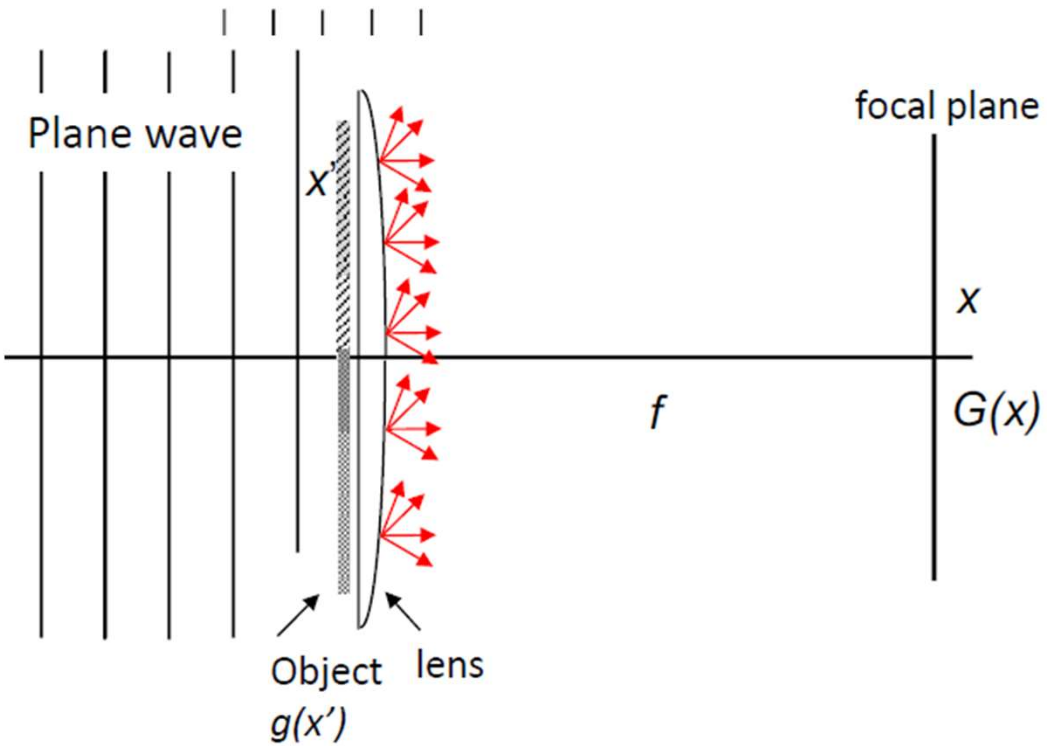
Diffraction limited systems

- Assume an ideal lens with aperture size D:



- Because of diffraction, a point source infinitely far away (a planar wave) will not be focused onto a single point in the image plane.

A lens produces the Fourier transform!



- We will show that $G(x)$ is the Fouriertransform of $g(x')$ (apart from a phase factor)
- Using Huygens' wave model for light.

Deriving the lens transform

- Add the light contribution from each point *x'* entering the lens to each point *x* in the focal plane, taking magnitude *A* and phase ϕ into account.
- Magnitude is given by the object density *g*(*x'*). Phase depends on the optical path length. Compute this separably for the lens and the path from the lens to the focal plane.
- Math trick: represent each light contribution by a complex number, $Ae^{j\phi}$, where *A* is the magnitude and ϕ is the phase relative to a common reference.
- 1D-analysis (can easily be extended to 2D).

Path length through the lens

Light rays passing through a lens are assumed to be close to the optical axis and at small angles with respect to it.

Use Taylor expansion

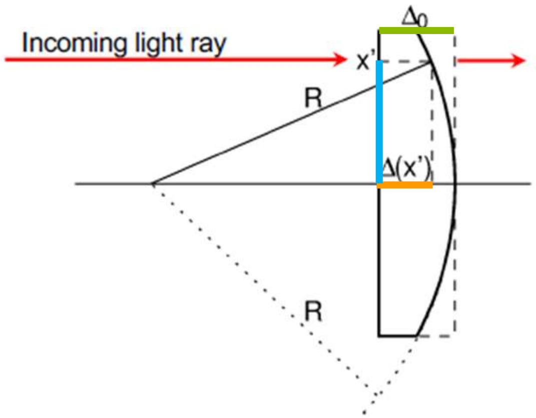
Simplifications

- The lens is plano-convex and thin
- Paraxial approximation
- Coherent light
- Inscribe the lens within a virtual rectangular box and apply Huygens' principle on the light coming out from this box.

$$\Delta(x') = \Delta_0 - \left(R - \sqrt{R^2 - x'^2}\right) \approx \Delta_0 - \frac{x'^2}{2R}$$

Where we have used x' << R (paraxial approximation)

Assume that the light travels slower by a factor *n* (*refractive index*) in the lens than in air and exits at the same height (x') since the lens is thin.



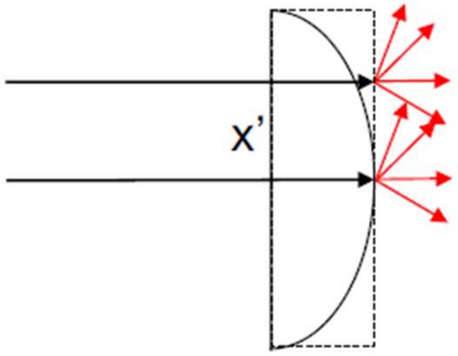
The "optical path length" travelled within the virtual box will then be

$$\delta(x') = n\Delta(x') + \left(\Delta_0 - \Delta(x')\right) = n\Delta_0 - (n-1)\frac{x'^2}{2R}$$

Applying "Lensmakers Formula" *1/f* = (*n*-1)/*R* gives:

$$\boxed{\delta(x') = n\Delta_0 - \frac{x'^2}{2f}}$$

The phase transform for the lens



The optical pathlength $\delta(x')$ corresponds to the phase shift:

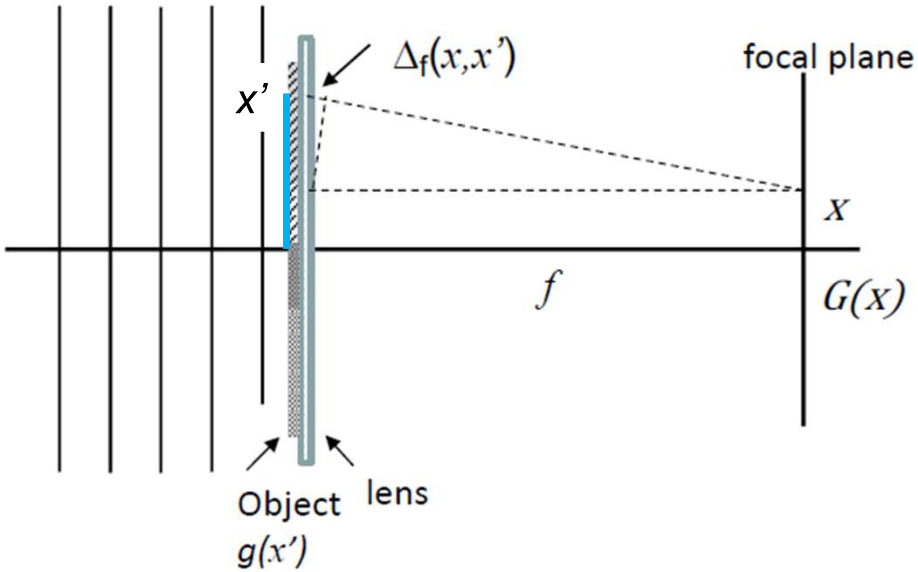
$$\Delta\phi = \frac{2\pi}{\lambda}\delta(x')$$

Inserting $\delta(x') = n\Delta_0 - \frac{x'^2}{2f}$

and disregarding the fixed delay $n\Delta_0$ gives the "phase transform":

$$\boxed{T_L(x') = e^{-j\frac{k}{2f}x'^2}} \quad \text{where} \quad k = \frac{2\pi}{\lambda}$$

The phase transform for from lens to focal plane



Use Taylor expansion

$$\Delta_f(x,x') = \sqrt{(x'-x)^2 + f^2} - f = \approx \frac{(x'-x)^2}{2f}$$

(again using the paraxial approximation)

Thus $\boxed{T_f = e^{j\frac{k}{2f}(x'-x)^2}}$

Putting it all together

$$G(x) = \int_{-\infty}^{\infty} g(x') \cdot T_L \cdot T_f dx' = \int_{-\infty}^{\infty} g(x') \cdot e^{-j\frac{k}{2f}x'^2} \cdot e^{j\frac{k}{2f}(x'-x)^2} dx' =$$

$$= e^{j\frac{k}{2f}x^2} \int_{-\infty}^{\infty} g(x') e^{-j\frac{k}{f}xx'} dx'$$

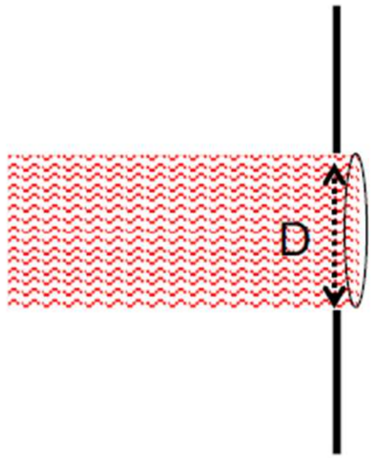
Normalise according to $u = x/(\lambda f)$

$$\boxed{\boldsymbol{G}(u) = e^{j\pi\lambda f u^2} \int_{-\infty}^{\infty} g(x') e^{-j2\pi u x'} dx'}$$

This is the Fourier Transform multiplied by a phase factor (of magnitude 1)!

See R. Forchheimer: Härledning av PSF för en tunn lins
on how to get rid of the phase factor by moving the object further away from the lens.

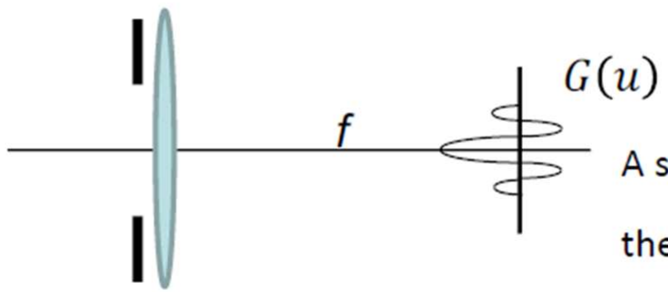
The effect of the aperture



The aperture can be viewed as an input image: $g(x') = rect(x'/D)$

The lens produces:

$$G(u) = \left(e^{j\pi\lambda f u^2}\right)\frac{\sin(\pi D u)}{\pi D u}$$



A screen at the image plane will show the (diffraction) pattern: $\left|\frac{\sin(\pi D u)}{\pi D u}\right|^2$

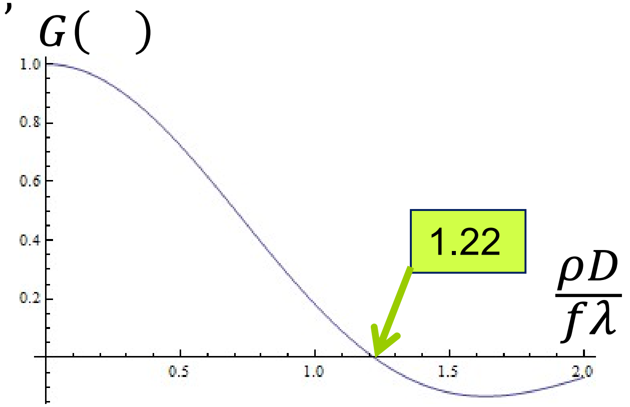
Diffraction limited systems

- $f(x) = \frac{\sin(\pi x)}{\pi x}$ is termed the "sinc(x)" function
- This phenomena generalizes to 2D:
  - The resulting wave-function $G(u, v)$ is the 2D Fourier transform of the incoming spatial amplitude $g(x', y')$
- Example: a circular aperture of diameter *D*
  - (Input amplitude normalized to $1/(f\lambda)$, $r = \sqrt{x'^2 + y'^2}$, $\rho = \sqrt{u^2 + v^2}$,)

$$\boxed{g(r) = \frac{1}{f\lambda} rect\left(\frac{r}{D}\right)}$$

First order Bessel function

$$\boxed{G(\rho) = \frac{J_1(\pi\rho D/(f\lambda))}{\pi\rho D/(f\lambda)}}$$

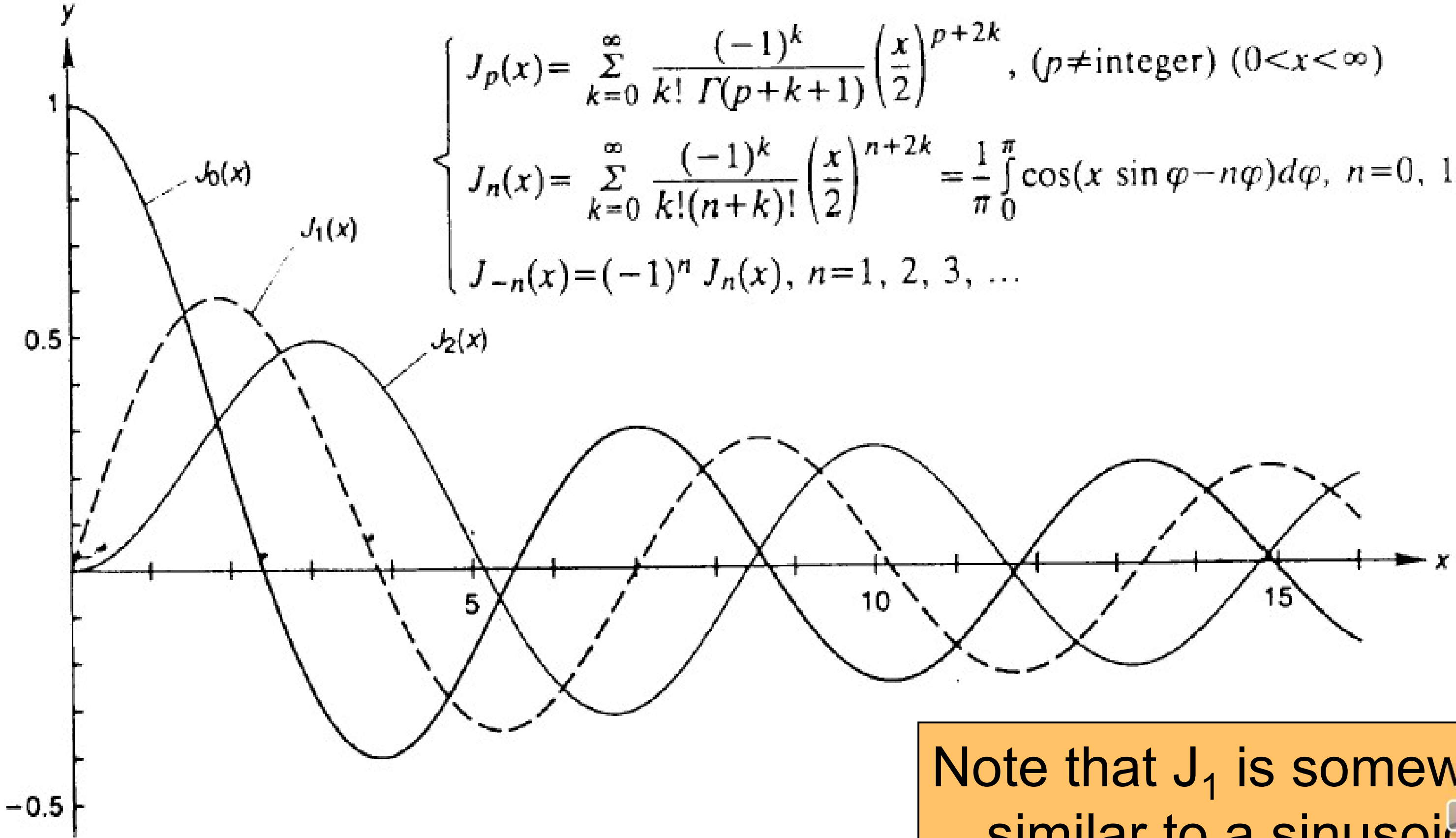


- G( ) is sometimes called the jinc function because it has similarities with the sinc function.

Bessel functions

Bessel functions $J_p(x)$

$$
\begin{cases}
J_p(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(p+k+1)} \left(\frac{x}{2}\right)^{p+2k}, \quad (p \neq \text{integer}) \; (0 < x < \infty) \\
J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!} \left(\frac{x}{2}\right)^{n+2k} = \frac{1}{\pi} \int_0^{\pi} \cos(x \sin\varphi - n\varphi)\, d\varphi, \quad n = 0, 1, 2, \ldots \\
J_{-n}(x) = (-1)^n J_n(x), \quad n = 1, 2, 3, \ldots
\end{cases}
$$

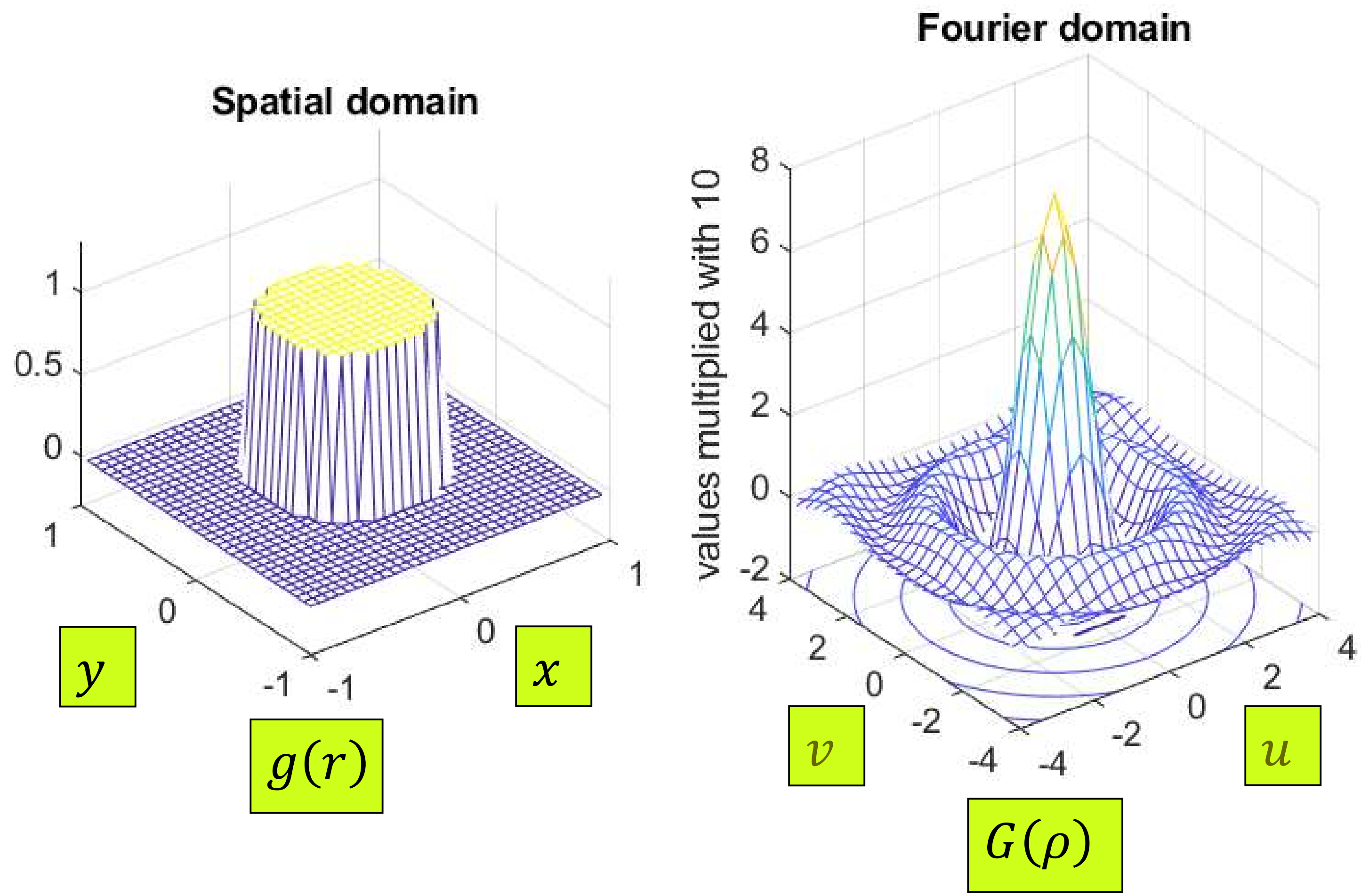


Note that J_1 is somewhat similar to a sinusoid.

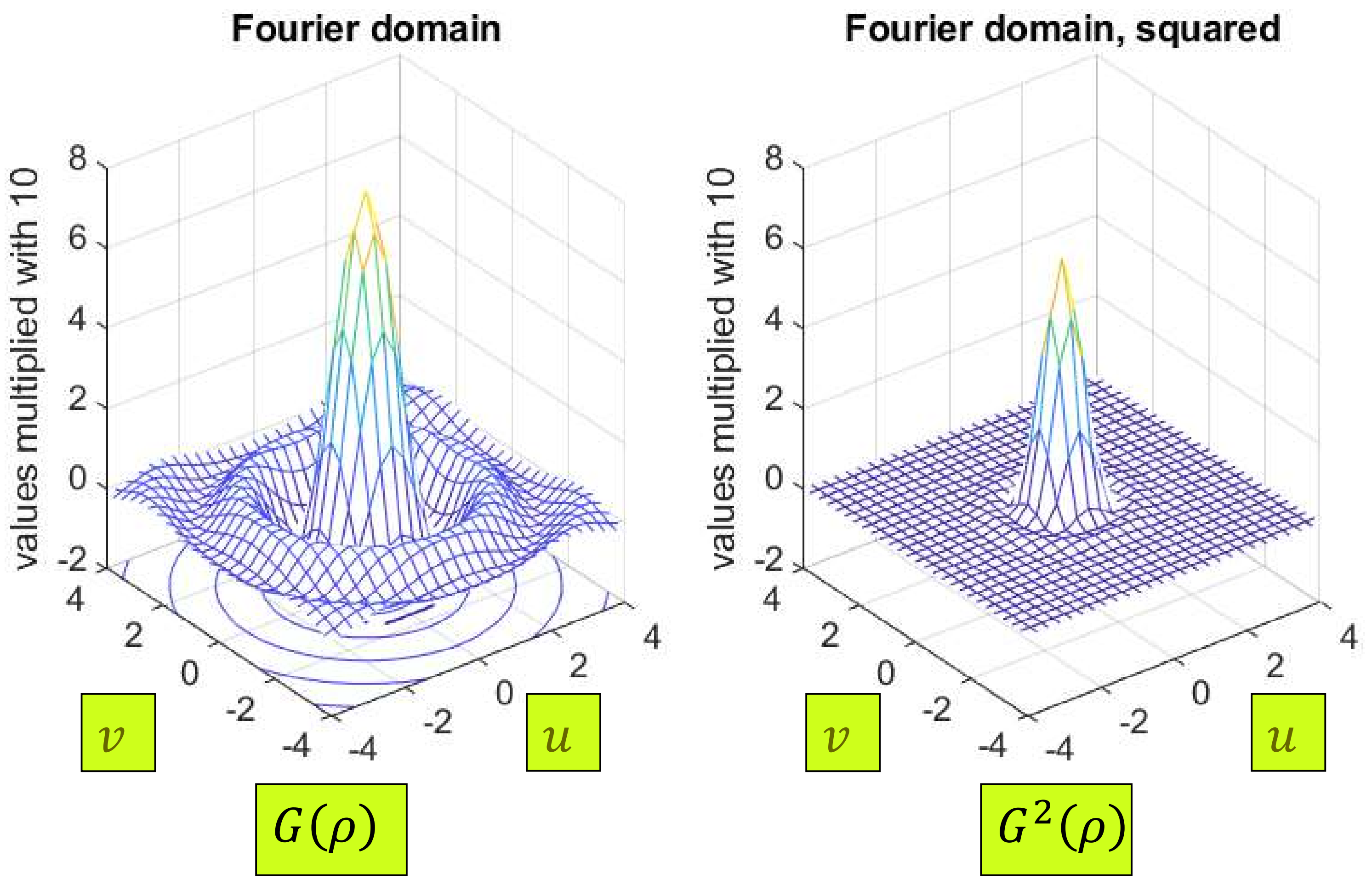
The Fourier transform of a circular disc

- The circular box $g(r)$ and its Fourier transform $G(\rho)$
  - $G(\rho)$ is sometimes called the *jinc* function because it has some similarities with the *sinc* function.
  - $G^2(k\rho)$ is sometimes called the *Airy* disk. It is the *point spread function* of a circular aperture with diameter D. This means that a point source at infinite distance (planar wave front) will give rise to an airy disk image when viewed through the aperture.

The circular box and its Fourier transform



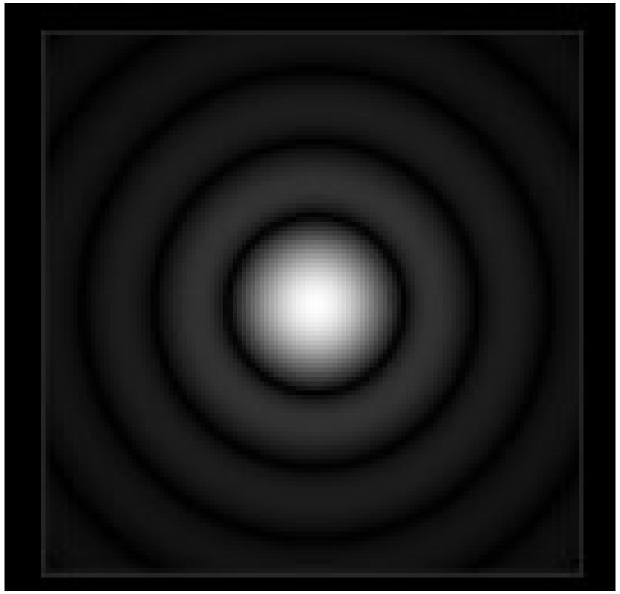
The Fourier transform and the squared Fourier transform



The Airy disk

- The image of a point-source for a diffraction-limited optical system is called *Airy pattern*. The central part is called the *Airy disk*.
  - Airy pattern: The image of a focused point-source becomes a diffraction pattern consisting of concentric light and dark circles.

- The distance from center to first dark ring is $1.22\ f\lambda/D$.
- The light intensity is given by the square of the jinc-function.



Resolution limit

- The smallest resolvable distance in the image plane, Δx, is given by

Distance to the first zero crossing in G( )

lens focal length

$$\Delta x \approx 1.22\,\lambda\,\frac{f}{D}$$

light wavelength

lens diameter

- The Rayleigh criterion for barely resolving two objects that are point sources of light, such as stars seen through a telescope, is that the center of the Airy disk for the first object occurs at the first minimum of the Airy disk of the second (same equation holds).

Resolution limit

Conclusions:

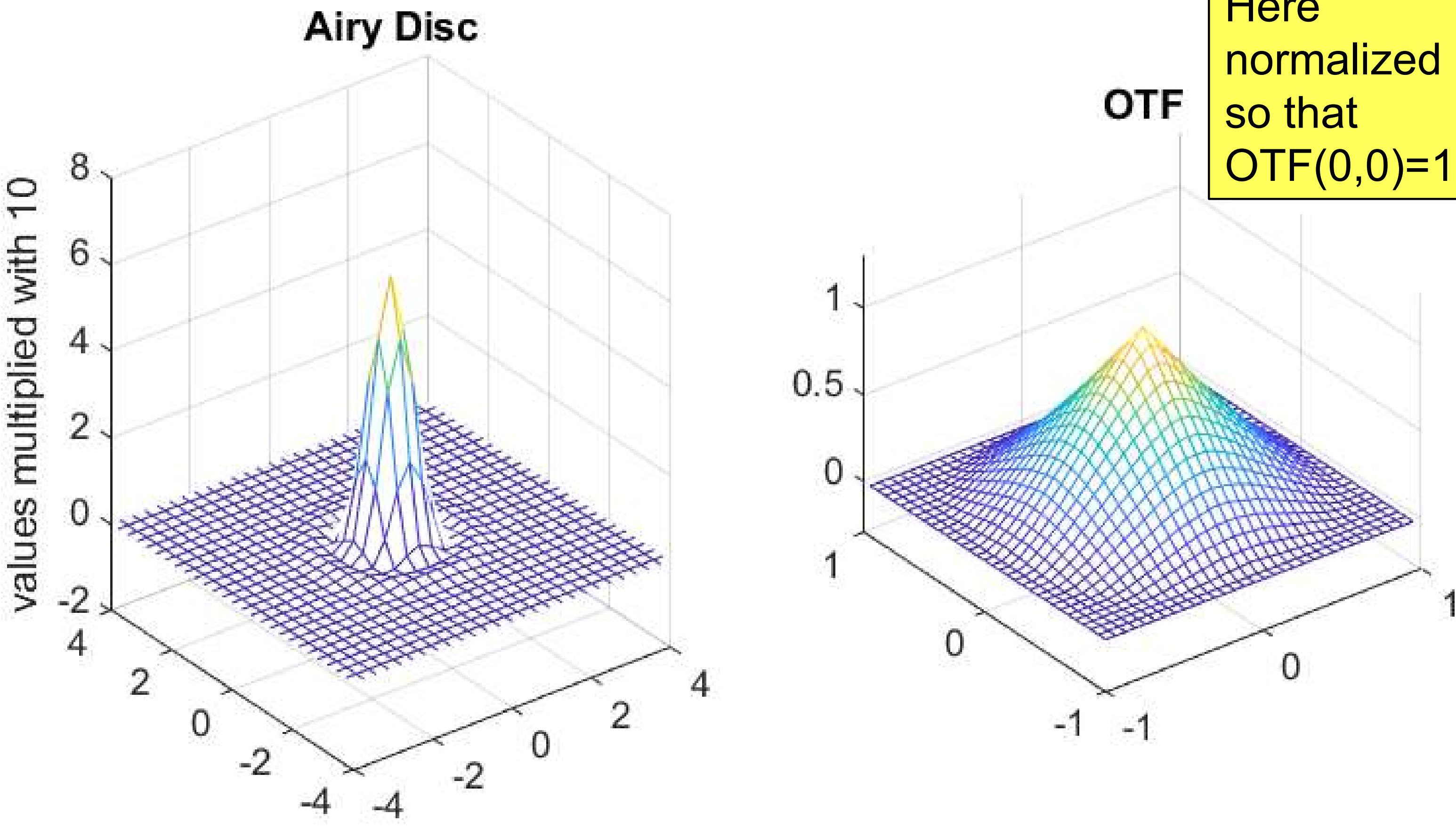
- The image cannot have a better resolution than Δx
- No need to measure the image with higher resolution than Δx !

- Be aware of cameras with high pixel resolution and high diffraction
  - Image resolution is not defined by number of pixels in the camera!

Optical transfer function (OTF)

- The optical transfer function (OTF) of an optical system specifies how different spatial frequencies are handled by the system.
- OTF is the Fourier transform of the PSF
- For an ideal lens system, in focus, the OTF is the Fourier transform of the Airy disk.
- Summary and observation
  - 1) The Fourier transform of the circular box is a jinc function.
  - 2) Similarly, the Fourier transform of the jinc function is a circular box.
  - 3) The Airy disk is the jinc function multiplied with itself.
  - 2) and 3) gives that the OTF, the Fourier transform of the Airy disk, is a circular box convolved with itself!

The Airy disc and its Fourier transform, the optical transfer function, OTF



Here normalized so that OTF(0,0)=1

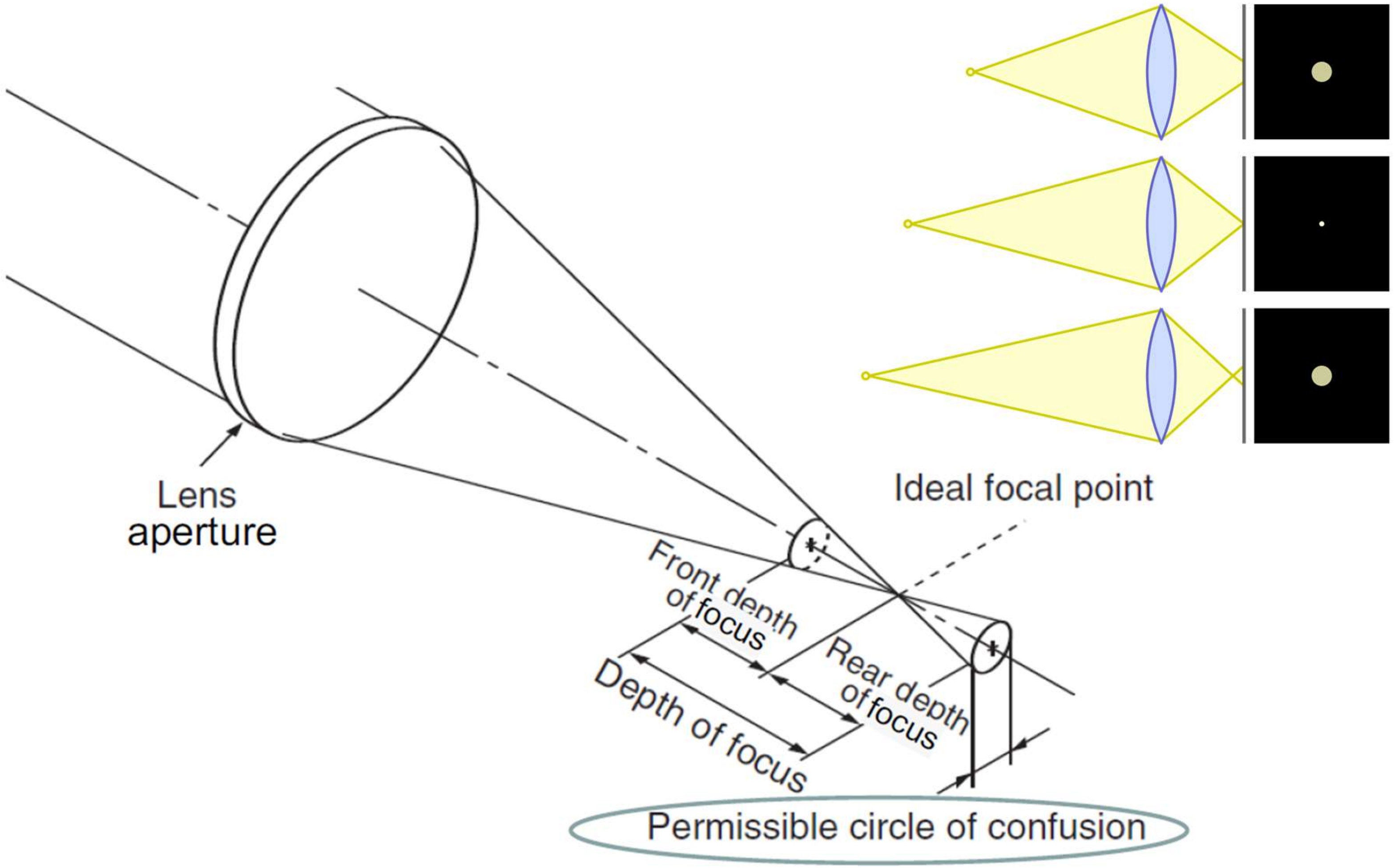
The point spread function (PSF)

- The **PSF** is a generalization of the point light source response
  - Diffraction: results in the Airy disk function
  - Out-of-focus blur: The out-of-focus PSF takes the shape of the camera aperture. For a circular aperture, the PSF is a disk, which is sometimes referred to as the **circle of confusion**.
- There are also other factors that contribute to the point spread function
  - Atmospheric turbulence
  - Optical aberrations
  - Motion
  - etc.
- The fact that the out-of-focus PSF takes the shape of the camera aperture is utilized for coded apertures, see the lecture on Specialized cameras.

Depth of field
"Skärpedjup" in Swedish

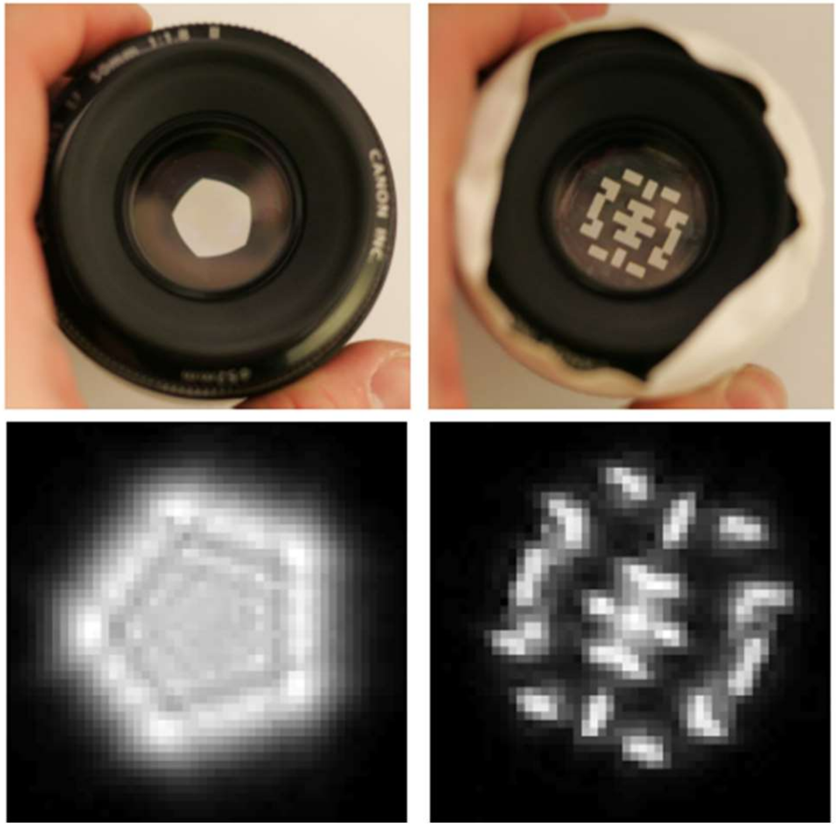
- The lens gives a focused image
  - Points that are off the object plane become blurred proportional to the displacement from the object plane
- Due to the resolution limit, it makes sense to accept blur in the order of Δx
  - This blur will be there anyway due to diffraction
- **Depth of field** (d) is the displacement along the optical axis from the object plane that gives blur $\leq \Delta x$

Depth of field, Depth of focus, Circle of confusion

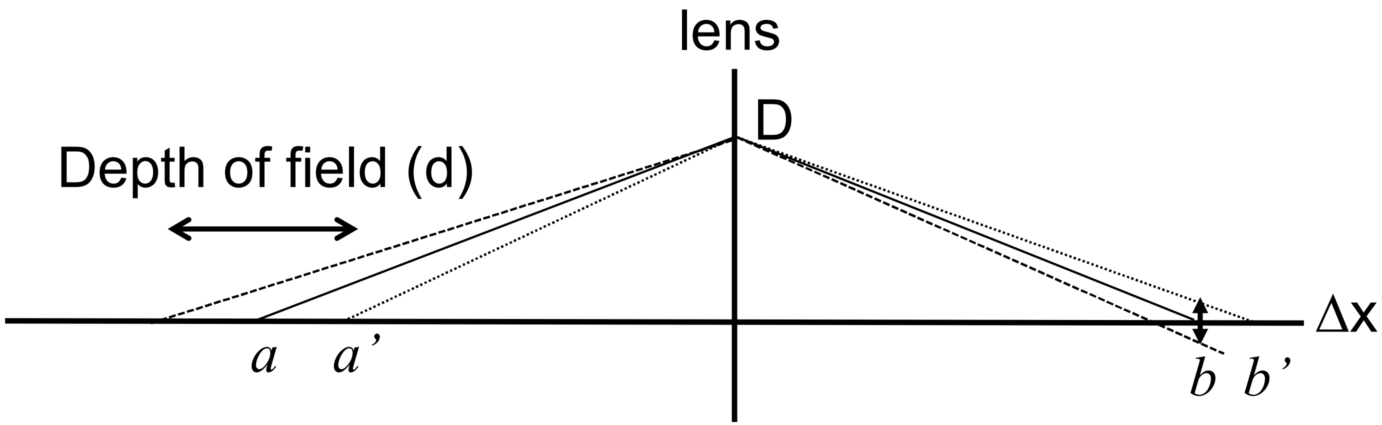


Examples of defocused images of a point source from other apertures than a circle

- Top left: a standard Canon 50mm f/1.8 lens with the aperture partially closed.
- Bottom left: the resulting blur pattern. The intersecting aperture blades give the pentagonal shape, while the small ripples are due to diffraction.
- Top right: the same model of lens but with a filter inserted into the aperture.
- Bottom right: the resulting blur pattern
- From Levin et al: Image and Depth from a Conventional Camera with a Coded Aperture



Depth of field



$$\frac{1}{a} + \frac{1}{b} = \frac{1}{f}$$

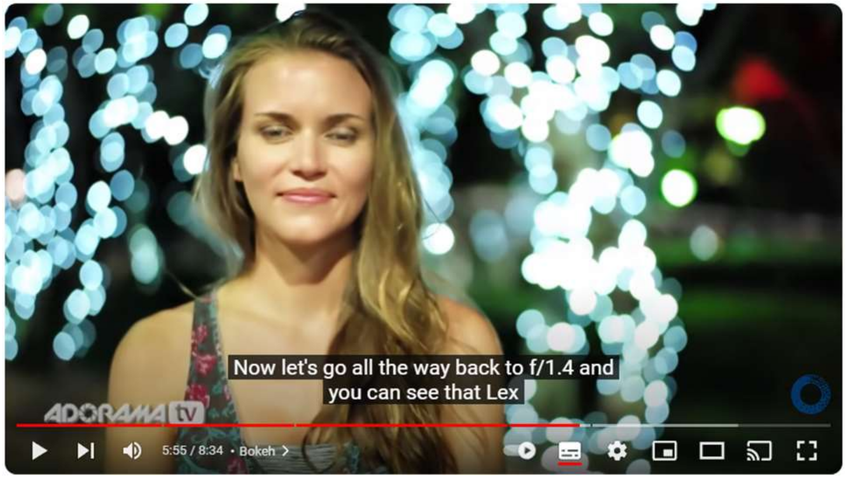
- Insert $a' = a$ - d/2 to get the horizontal blur (b'-b)
- Relate the horizontal blur to the vertical blur Δx

Depth of field, image example

- Blur both in front of and behind the person of interest, who is in the object plane.

Circle of confusion, image example

- f/5.6 is a smaller aperture than f/1.4
- Top image: Smaller aperture gives less light and larger depth of field.
- Bottom image: Larger aperture gives more light and a smaller depth of field. The woman is in focus, but the point light sources are defocused, giving visible circles of confusion.
- https://www.youtube.com/watch?v=eJHIVR4_dEE&t=2s
- Another nice video:
- https://www.youtube.com/watch?v=Pdq65IEYFOM



Depth of field, equations

- For a camera where $a < \infty$, an approximation(assuming $d \ll a$) for *d* is

$$\boxed{d \approx 2\Delta x \frac{a(a-f)}{Df}}$$

  - a = distance from lens to the object plane
  - *f* = lens focal length
  - *D* = lens diameter
  - Δx = required image plane resolution
  - *d* = depth of field

Depth of field, equations

- For a camera where $a = \infty$, points that are further away than d_{min} are blurred less than Δx, where

$$\boxed{d_{min} = \frac{fD}{4\Delta x}}$$

The F-number (a photography term)

- f/D is the **F-number** of the lens or lens system

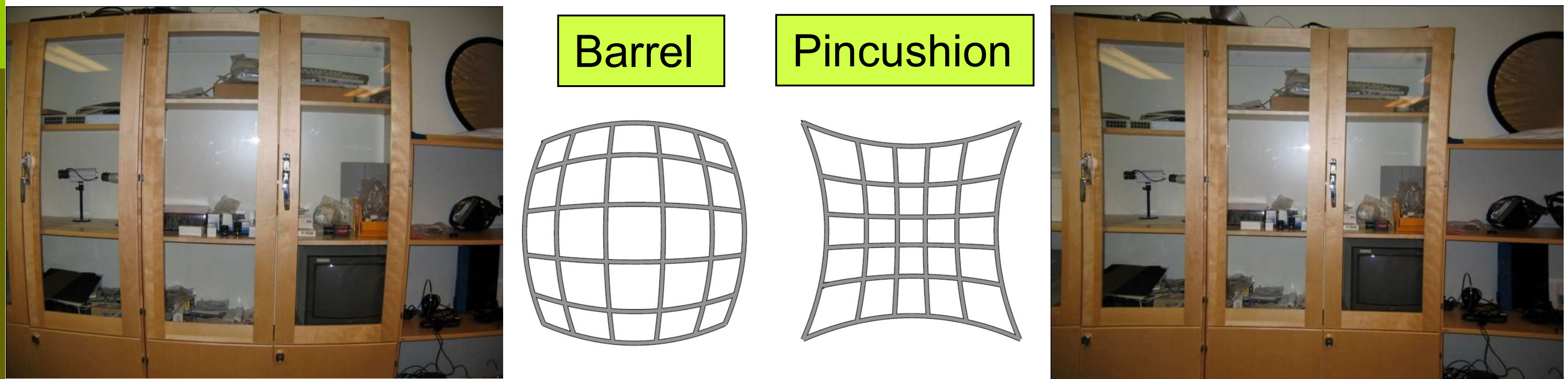
- Example
  - A typical **F-number** of a camera = 8
  - Blue light = 420 nm wavelength
  - Airy disk diameter $\Delta x = 1.22\ \lambda\ F \approx 4\ \mu m$

- For a lens with $f = 15$ mm we get
  - $d \approx 0.6$ m at $a = 1.5$ m
  - $d_{min} \approx 1.8$ m at $a = \infty$

This means that the depth of field is within a manageable range

Lens distortion

- A lens or a lens system can never map straight lines in the 3D scene exactly to straight lines in the image plane
- Depending on the lens type, a square pattern will typically appear like a **barrel** or a **pincushion**
- We will talk more about lens distortion in the Camera Calibration lectures,

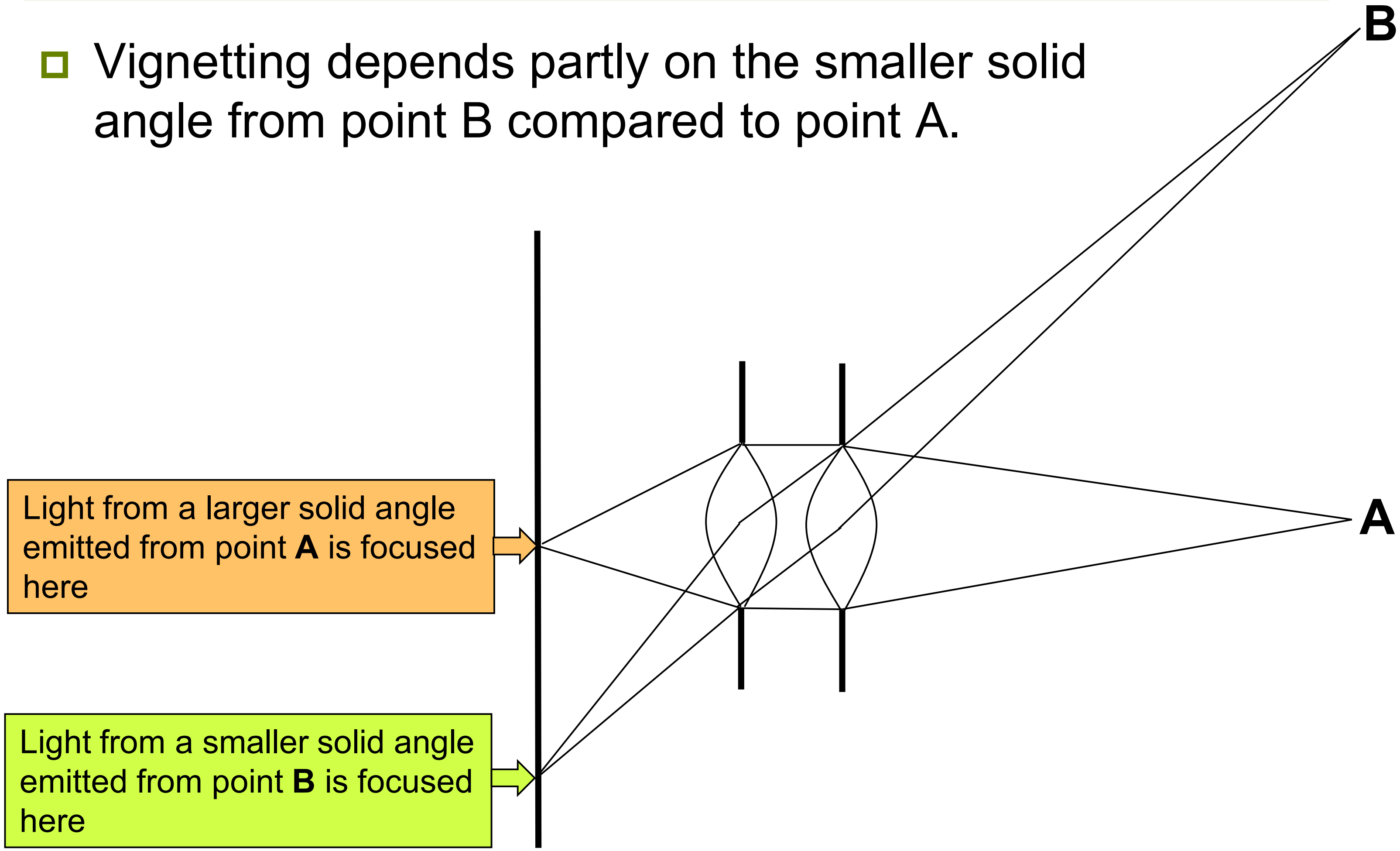


Vignetting

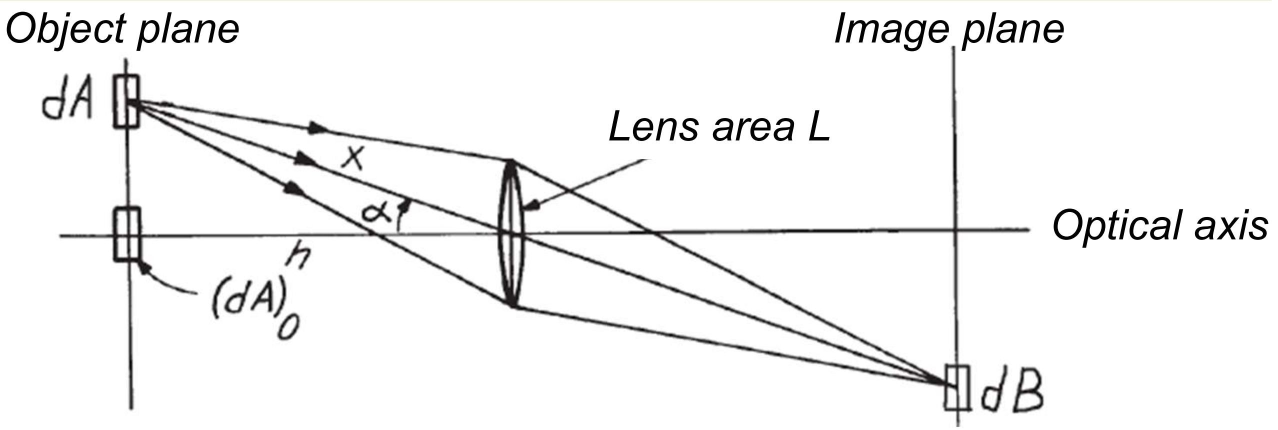
- Even if the light that enters the camera is constant in all directions, the image plane will receive a different amount of illumination. This effect is called **vignetting**.
- The attenuation of the image towards the edges of the image is approximately according to $\mathbf{cos^4}\alpha$, where α is the angle to the optical axis.
- Sometimes used as a photographic effect, but usually unwanted.
- Can be compensated for in digital cameras, by using a shading correction technique.

Vignetting

- Vignetting depends partly on the smaller solid angle from point B compared to point A.



Derivation of the $\cos^4$ law



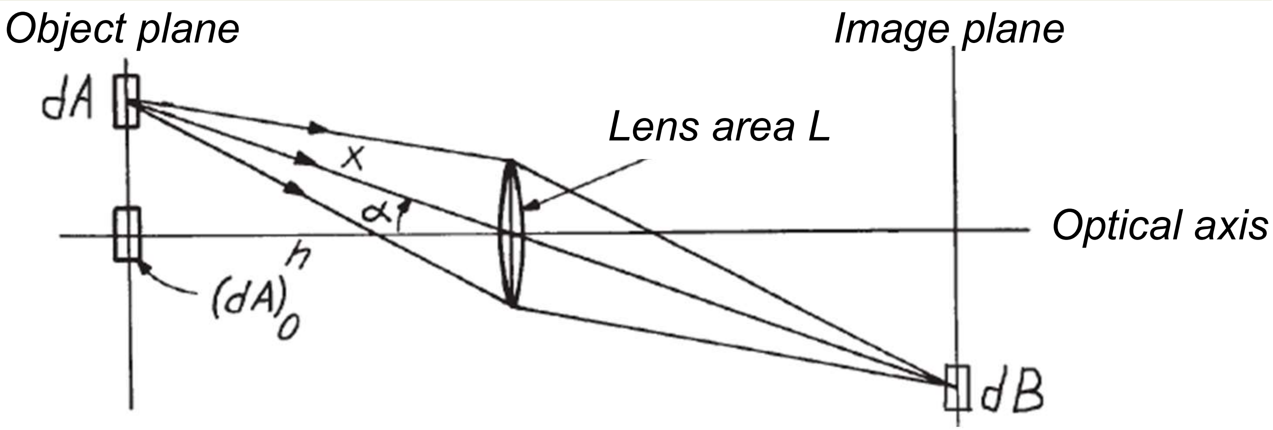
- A surface element *dA* in the object plane is mapped through the lens onto a surface element *dB* in the image plane.
- A surface element *dA* in the object plane is placed at the distance:

  $x = h/\cos\alpha$
- The size of the lens area in the direction towards the surface element is:

  $L\cos\alpha$
- The surface element thus irradiates the lens with the solid angle:

  $$\frac{L\cos\alpha}{x^2} = \frac{L\cos^3\alpha}{h^2}$$

Derivation of the $\cos^4$ law, cont.



- A surface element in the image plane with area dB has this surface directed towards the lens:

  $dB \cos \alpha$

- The distance between the lens and image element does not affect the brightness, because the lens refracts the light towards the image plane.
- Combining the two last formulas finally gives that the brightness being proportional to:

$$\frac{L \cos^3 \alpha}{h^2} \cdot dB \cos \alpha = \frac{dB \cdot L \cos^4 \alpha}{h^2}$$

Proportional to $\cos^4$!

Chromatic aberration

- The refraction index of matter (lenses) is wavelength dependent
  - Example: a prism can decompose the light into its spectrum

  - A ray of white light is decomposed into rays of different colors that intersect the image plane at different points

Chromatic aberration

- Sometimes clearly visible if you look close to the edges through a pair of glasses