TSBB21, Lecture 9

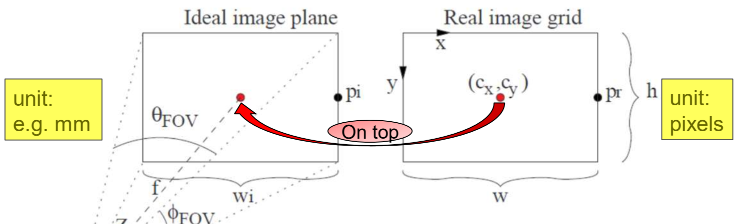
Panorama Stitching. Mathematical tool: SVD

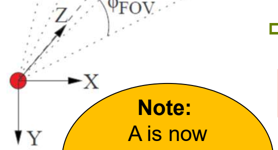
- Panorama Stitching
- Mathematical tool: SVD

- Literature
  - "Panorama Stitching. Supplementary Notes" by Per-Erik Forssén
  - Last in … "Short about camera geometry and camera calibration" by Maria Magnusson
  - Parts of … "Introduction to Representations and Estimation in Geometry" (IREG) by Klas Nordberg
  - Parts of … "Mathematical Toolbox for Studies in Visual Computation at Linköping University" by Klas Nordberg

Panorama stitching

Determination of FOV and offset





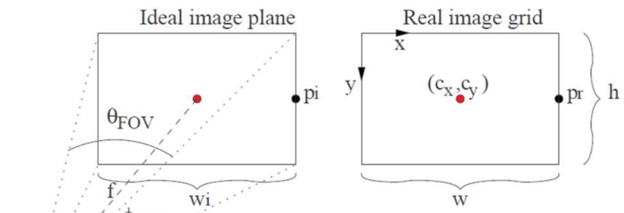
- Let the world coordinate system and the camera coordinate system be aligned:

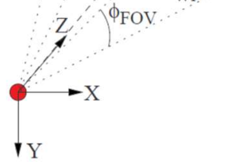
$$\tilde{\mathbf{x}} \sim \mathbf{K}\,\mathbf{X} \qquad (x, y, 1)^T \sim \mathbf{K} \cdot (X, Y, Z)^T$$

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \sim \begin{pmatrix} xZ \\ yZ \\ Z \end{pmatrix} = \begin{pmatrix} K_{11} & K_{12} & c_x \\ 0 & K_{22} & c_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

Note:
A is now denoted K
(u,v) is now denoted (x,y)

Offset between optical centre and geometrical centre

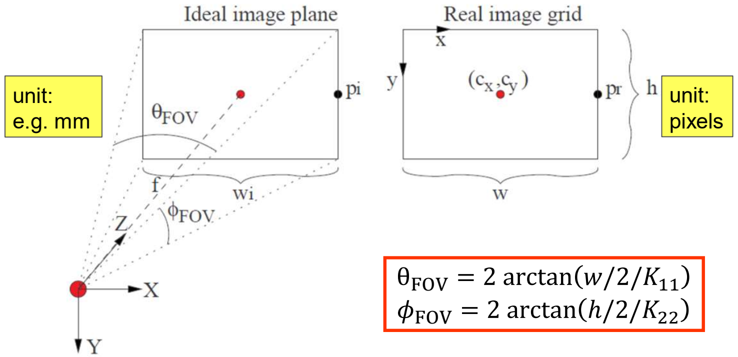




$$\mathrm{horiz}_{\mathrm{off}} = \arctan\left((w/2 - c_x)/K_{11}\right)$$
$$\mathrm{vert}_{\mathrm{off}} = \arctan\left(\left(h/2 - c_y\right)/K_{22}\right)$$

Note:
These angles are normally small.

Determination of Field Of View (FOV)



$$\theta_{\mathrm{FOV}} = 2\arctan(w/2/K_{11})$$
$$\phi_{\mathrm{FOV}} = 2\arctan(h/2/K_{22})$$

- Proof: Panorama Stitching. Supplementary Notes.
- Also in next slide.

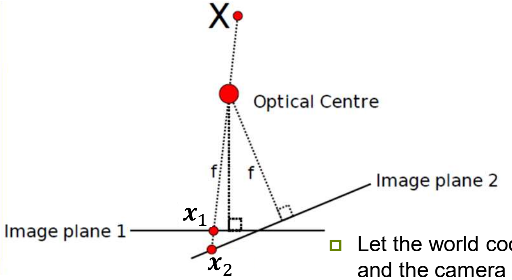
Determination of Field Of View (FOV), proof

See figure 1. For simplicity, assume that $\mathrm{horiz}_{\mathrm{off}} = \mathrm{vert}_{\mathrm{off}} = 0$. Then the points $p_i = (w_i/2, 0, f)^T$ and $p_r = (c_x + w/2, c_y, 1)^T$. Therefore

$$Z\begin{pmatrix} c_x + w/2 \\ c_y \\ 1 \end{pmatrix} = \begin{pmatrix} K_{11} & K_{12} & c_x \\ 0 & K_{22} & c_y \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} w_i/2 \\ 0 \\ f \end{pmatrix}$$

The first row gives $Z(c_x + w/2) = K_{11}w_i/2 + c_x f$ and the third row gives $Z = f$. Therefore $f(c_x + w/2) = K_{11}w_i/2 + c_x f$, which gives $w_i = wf/K_{11}$. Finally, $\theta_{\mathrm{FOV}} = 2\arctan(w_i/(2f)) = 2\arctan(w/(2K_{11}))$. Similarly, $\phi_{\mathrm{FOV}} = 2\arctan(h/(2K_{22}))$.

Rotational Homographies



- Let the world coordinate system origin and the camera origin be aligned:

$$\boldsymbol{x} = (x, y)^T$$

$$\tilde{\mathbf{x}} \sim \mathbf{K}\,\mathbf{R}\,\mathbf{X} \qquad (x, y, 1)^T \sim \mathbf{K}\,\mathbf{R}\cdot(X, Y, Z)^T$$

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \sim \begin{pmatrix} K_{11} & K_{12} & c_x \\ 0 & K_{22} & c_y \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

Rotational Homographies

- The projections of a point **X** in the world to the points $\mathbf{x}_1$ and $\mathbf{x}_2$ in the two images:

$$\tilde{\mathbf{x}}_1 \sim \mathbf{K}\,\mathbf{R}_1\,\mathbf{X} \qquad \tilde{\mathbf{x}}_2 \sim \mathbf{K}\,\mathbf{R}_2\,\mathbf{X}$$

- Assume the existence of a homography $\mathbf{H}_{21}$ that maps points from image2 to image1:

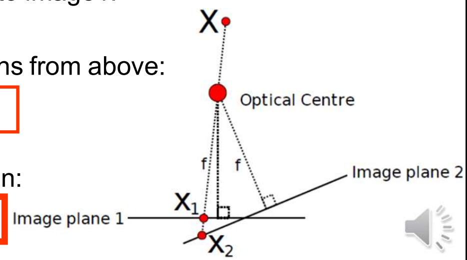
$$\tilde{\mathbf{x}}_1 \sim \mathbf{H}_{21}\tilde{\mathbf{x}}_2$$

- Insert the expressions from above:

$$\mathbf{K}\,\mathbf{R}_1\,\mathbf{X} \sim \mathbf{H}_{21}\,\mathbf{K}\,\mathbf{R}_2\,\mathbf{X}$$

- This is satisfied when:

$$\mathbf{H}_{21} = \mathbf{K}\,\mathbf{R}_1\,\mathbf{R}_2^{\mathrm{T}}\,\mathbf{K}^{-1}$$



Panorama stitching

- In panorama stitching, we have a set of images that share a common camera centre (origin), i.e. the images are all taken **from the same view-point** but in different directions.
- Given that the objects in the images are far away, the camera centres do not have to be exactly at the same point.
- Each image can be transformed into any other by a homography.

Panorama stitching

- By applying the homography $\mathbf{H}_{21}$ to image2 (which is taken by camera2), it can the **stitched** onto image1 (which is taken by camera1).
- By applying the homography $\mathbf{H}_{31}$ to image3 (which is taken by camera3), it can the **stitched** onto image1 (which is taken by camera1).
- If both image2 and image3 are stitched onto image1, image1 works as a **reference image**.
- It is possible to stitch a whole set of images onto one reference image.

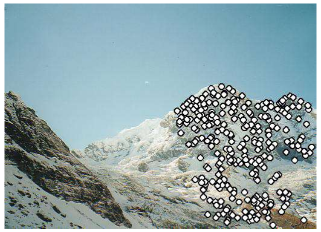
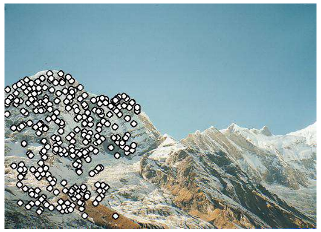
Panorama stitching, example

- Two images taken from approximately the same view-point:


Images from: *Automatic Panoramic Image Stitching using Invariant Features*, IJCV 2007, Matthew Brown

Panorama stitching, example

- Mark a set of corresponding points:

- From these points: Estimate a homography **H** that relates the 2 images.

Panorama stitching, example

- The right image stitched onto the left image:

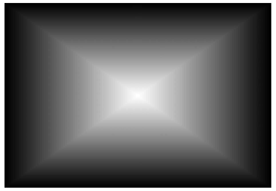
Practical issues

- The pixel values in overlapping regions may differ even if the geometric transformation is correct
  - Vignetting effects
  - Interpolation effects
  - Exposure time or illumination may be different in two images
  - Moving objects in the scene
- At each pixel:
  - Alternative 1: Take the value from only one of the two images
  - Alternative 2: Blend

Blending weight

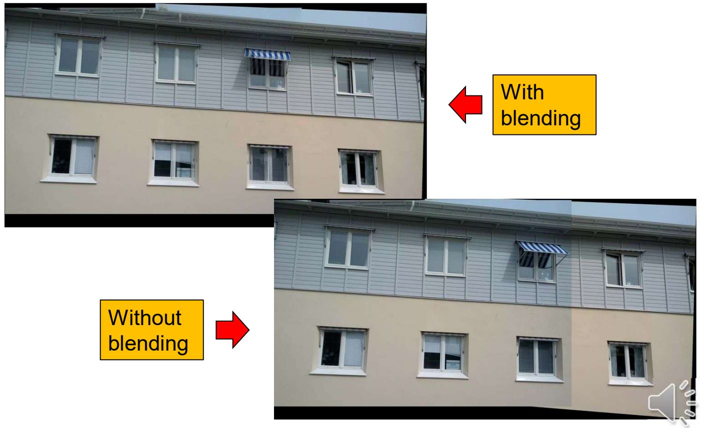
- For example, use a weight that is smaller at the edges of the image and larger at the center:

This is the weight image before the homography transformation



- Pano1 and Pano2 are the two images transformed to the reference grid.
- alpha1 and alpha2 are the weight image transformed to the reference grid.
- Normalized weighting: $\boxed{\text{Pano} = \frac{\text{alpha1} \cdot \text{Pano1} + \text{alpha2} \cdot \text{Pano2}}{\text{alpha1} + \text{alpha2}}}$

Blending



With blending

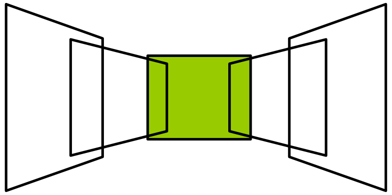
Without blending

Practical issues

- To assure a homography between any pair of images, we must have a pin-hole camera
  - No significant amount of lens distortion is allowed
  - Alternatively: lens distortion can be estimated and compensated for before the stitching
- In the panorama computer exercise, we use the following radial distortion model, $\boxed{r_{\text{out}} = \frac{\arctan(r_{\text{in}} \cdot \gamma)}{\gamma}}$
  - where the image is described in polar coordinates, (r, θ).
  - You will manually estimate γ by undistorting the images using several different γ -values. γ is good when straight lines in the world give straight lines in the image.
  - γ is small, e.g. γ=0.001

Practical issues

- If the view direction between the reference image and the stitched image is very different, the 'resolution in' and the 'size of' the two images will vary a lot.
- Ex) 4 images stitched to a reference image (green):



- An attempt to stitch an image from 90º view direction onto an image from 0º view direction will result in infinity size of the stitched image.
- Solution: Map the images onto a cylinder or sphere and stitch them there instead.

Mapping a point $(X,Y,Z)^T$ to the unit sphere

- A point $(X,Y,Z)^T$ is projected to the normalized image plane $(x_n,y_n,1)^T$ and then transformed to a point $(x,y,1)^T$ on the real image grid as:

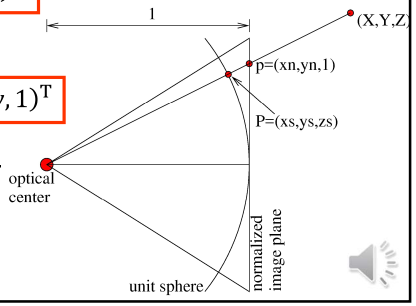
$$\boxed{(x, y, 1)^{\mathrm{T}} = \mathbf{K} \cdot (x_n, y_n, 1)^{\mathrm{T}}}$$

- Consequently:

$$\boxed{(x_n, y_n, 1)^{\mathrm{T}} = \mathbf{K}^{-1} \cdot (x, y, 1)^{\mathrm{T}}}$$

- Finally, a simple geometrical consideration gives:

$$\boxed{(x_s, y_s, z_s)^{\mathrm{T}} = \cdots}$$



The Orthogonal Procrustes problem (OPP)

- If two corresponding sets of 3D points from two images are mapped to the unit sphere, it is possible to determine the rotation between them using OPP.
- Consider two sets of 3D points $\boxed{\mathbf{X} = (\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_N)}$ and $\boxed{\mathbf{Y} = (\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_N)}$ that are related as: $\boxed{\mathbf{X}_k = \mathbf{R}\mathbf{Y}_k + \epsilon_k,}$
- where **R** is an orthogonal matrix and ϵ_k is an additive Gaussian noise term.
- It can be shown (see e.g. Panorama Stitching. Supplementary Notes) that
  if the SVD of $\mathbf{X}\mathbf{Y}^{\mathrm{T}}$ gives: $\boxed{\mathbf{X}\mathbf{Y}^{\mathrm{T}} = \mathbf{U}\mathbf{D}\mathbf{V}^{\mathrm{T}}}$
  Then: $\boxed{\mathbf{R} = \mathbf{U}\mathbf{V}^{\mathrm{T}}}$

Axis-Angle Representation

- When **R** is determined, it is possible to find the rotation axis $\hat{\mathbf{n}}$ and the rotation angle $\alpha \in [0, \pi[$
- $\hat{\mathbf{n}}$ is an eigenvector of R with eigenvalue 1:

  $\boxed{\hat{\mathbf{n}} = \mathbf{R}\,\hat{\mathbf{n}}}$
- The other two eigenvalues are:

  $\boxed{e^{i\alpha}}$ $\boxed{e^{-i\alpha}}$

- Use the Matlab command [V D] = eig(R);
- The eigenvectors are in V and the eigenvalues are in D.
- *More information on this can be found in e.g. Panorama Stitching. Supplementary…*

Resampling to Spherical Coordinates

- Now we have computed different **R** matrices for each image to be stitched. They can then be resampled to spherical coordinates:

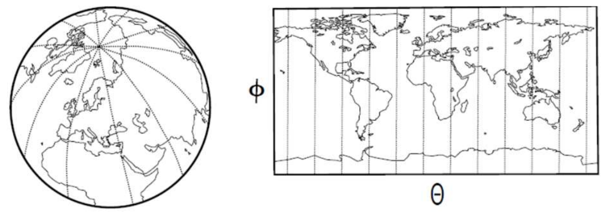
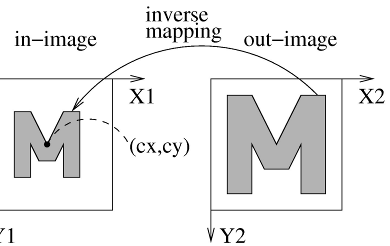


Fig. Illustration of spherical coordinates. Left: A world map painted on a sphere. Right: The same map in longitude-latitude space.

Resampling and interpolation
from in-image Im1 to out-image Im2

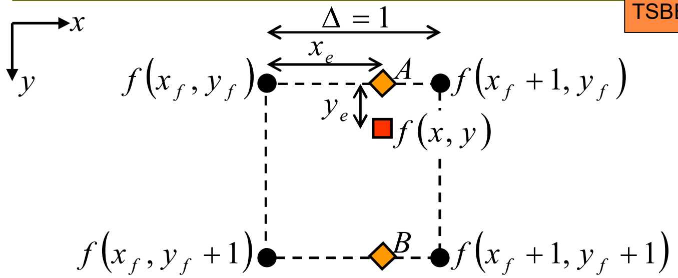
Repetition from TSBB08, TSBB31

- Suppose $(X2,Y2)^T = \mathbf{T} \cdot (X1,Y1)^T$
- For all points (X2,Y2) in the out-image:
  - Perform an inverse mapping inv(**T**) to (X1,Y1) in the in-image.
  - Perform interpolation, e.g. bilinear interpolation, in the in-image, obtain a value.
  - Put the value at position (X2,Y2) in the out-image



Bilinear interpolation

Repetition from TSBB08, TSBB31



$$\begin{cases} A = f(x_f, y_f) \cdot (1 - x_e) + f(x_f + 1, y_f) \cdot x_e \\ B = f(x_f, y_f + 1) \cdot (1 - x_e) + f(x_f + 1, y_f + 1) \cdot x_e \\ f(x, y) = A \cdot (1 - y_e) + B \cdot y_e \end{cases}$$

Spherical coordinate system: standard form

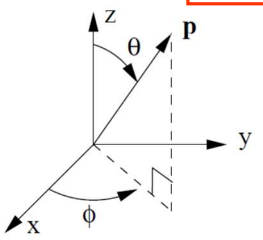
- Disadvantage: Singularities at

$$\theta = \{0, \pi\}$$

Image here

$$\boxed{\boldsymbol{p} = (x, y, z)^{\mathrm{T}}}$$

$$\boxed{r^2 = x^2 + y^2 + z^2}$$

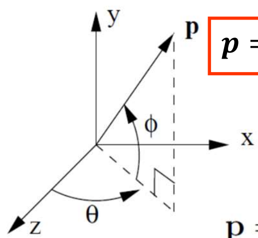


$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r \sin\theta \cos\phi \\ r \sin\theta \sin\phi \\ r \cos\theta \end{pmatrix}$$

Spherical coordinate system: longitude-latitude form

- Singularities at $\phi = \left\{-\frac{\pi}{2}, \frac{\pi}{2}\right\}$

i.e. above and below the camera, north and south pole.

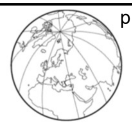


$$\boxed{\boldsymbol{p} = (x, y, z)^{\mathrm{T}}}$$

$$\boxed{r^2 = x^2 + y^2 + z^2}$$

$$\mathbf{p} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r \cos\phi \sin\theta \\ r \sin\phi \\ r \cos\phi \cos\theta \end{pmatrix}$$

Image here

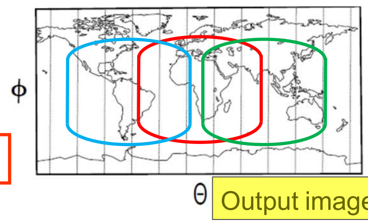


Recipe

- For all points (ϕ, θ) in the output image:
  - Transform it with

$$\boxed{(sx, sy, s)^T = \tilde{\mathbf{x}} \sim \mathbf{KRp}(\phi, \theta)}$$

  - Receive the position (x, y) in the input image.
  - Perform bilinear interpolation in the input image, obtain a value.
  - Put the value at position (ϕ, θ) in the output image



Output image

Note that, normally, there are several input images, e.g. three. The reference image (**red**) have the rotation **R=I**, the identity matrix. The other images (**blue** and **green**) have rotations $\mathbf{R}=\mathbf{R}_1$ and $\mathbf{R}=\mathbf{R}_2$, relative to the reference image.

Singular value decomposition (SVD)

Theorem:

- For **any** N×M real valued matrix **X**,
  - we can find an N×N orthonormal matrix **U**
  - we can find an M×M orthonormal matrix **V**
  - we can find an N×M real diagonal matrix **S**
  - such that:

$$\boxed{\mathbf{X} = \mathbf{USV}^T}$$

  - This is the **singular value decomposition (SVD)** of **X**

Singular value decomposition (SVD)

- **S** is N×M diagonal (non-zero values only in the diagonal)
- The diagonal elements of **S** , $\sigma_1, \ldots, \sigma_P$, are **real** and **non-negative** (with P = min(N, M))
- The diagonal elements of **S** are the **singular values** of **X**
- The singular values are usually ordered such that $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_P$

Singular value decomposition (SVD)

- For singular value σ_k, the corresponding columns $\mathbf{u}_k$ and $\mathbf{v}_k$ of **U** and **V** are the **left and right singular vectors** of **X**, respectively.
- Notice that $\boxed{\begin{aligned}\mathbf{X}\mathbf{v}_k &= \sigma_k \mathbf{u}_k \\ \mathbf{X}^{\mathrm{T}}\,\mathbf{u}_k &= \sigma_k \mathbf{v}_k\end{aligned}}$
- Remember that $\boxed{\mathbf{X} = \mathbf{USV}^T}$

Singular value decomposition (SVD)

- In the case of a non-square matrix **X** there will be some left (or right) singular vectors that neither have a corresponding singular value σ_k nor a right (or a left) singular vector.
- In this case they are simply said to have singular value 0 since, for example

$\mathbf{X}\,\mathbf{v}_k = \mathbf{0}$ (k > P, and N < M)

Solution of a homogeneous system of equations using SVD

- Regard the following homogeneous system of equations:

$$\boxed{\mathbf{Xb} = 0}$$

- Perform SVD and rewrite:

$$\boxed{\begin{aligned}\mathbf{USV}^{\mathrm{T}}\mathbf{b} &= 0 \\ \mathbf{U}^{\mathrm{T}}\mathbf{USV}^{\mathrm{T}}\mathbf{b} &= 0 \\ \mathbf{SV}^{\mathrm{T}}\mathbf{b} &= 0\end{aligned}}$$

- Solution:

$$\boxed{\mathbf{b} = \mu \mathbf{v}_n}$$

- where μ is a scale factor and $\mathbf{v}_n$ is the last column of **V**
- Matlab code:

```
[U,S,V] = svd(X);
b = V(:,n);
```

Example: Solve $\boxed{\mathbf{Xb} = 0}$

- Suppose that **X** is a 4 × 3-matrix. Since $\mathbf{X} = \mathbf{USV}^\mathrm{T}$, we have:

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \\ x_{41} & x_{42} & x_{43} \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ u_{21} & u_{22} & u_{23} & u_{24} \\ u_{31} & u_{32} & u_{33} & u_{34} \\ u_{41} & u_{42} & u_{43} & u_{44} \end{pmatrix} \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_{11} & v_{21} & v_{31} \\ v_{12} & v_{22} & v_{32} \\ v_{13} & v_{23} & v_{33} \end{pmatrix}$$

- Previous slide gave: $\boxed{\mathbf{SV}^\mathrm{T}\mathbf{b} = 0}$
- Solution: $\boxed{\mathbf{b} = \mu\mathbf{v}_3 = \mu(v_{13}, v_{23}, v_{33})^\mathrm{T}}$ Insert:

$$\begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_{11} & v_{21} & v_{31} \\ v_{12} & v_{22} & v_{32} \\ v_{13} & v_{23} & v_{33} \end{pmatrix} \mu \begin{pmatrix} v_{13} \\ v_{23} \\ v_{33} \end{pmatrix} = \mu \begin{pmatrix} 0 \\ 0 \\ \sigma_3 \end{pmatrix}$$

- The smaller σ_3, the better solution and $\sigma_3 = 0$ solves **Xb** = 0 perfectly.