

TSBB21, Lecture 8

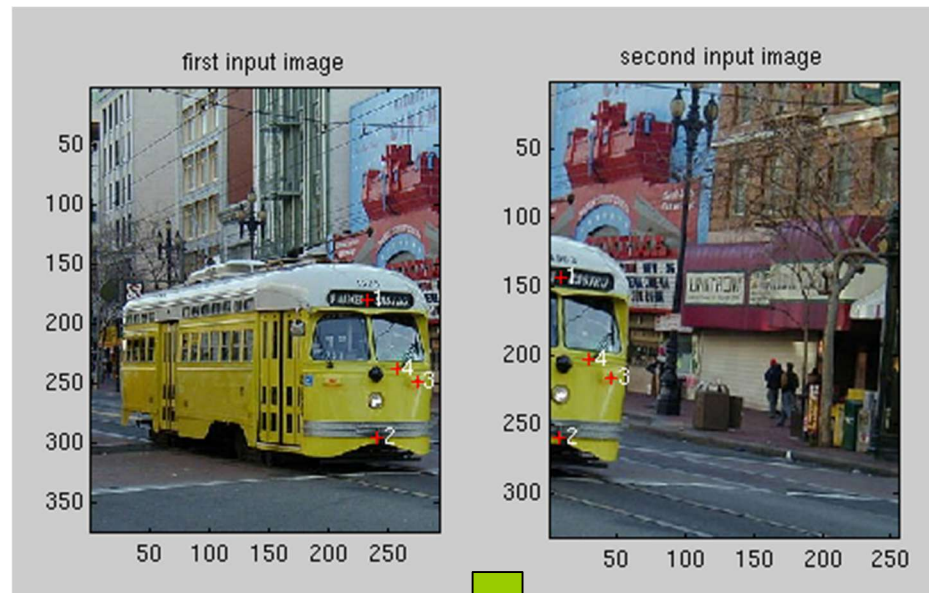
Panorama Stitching. Mathematical tool: SVD

- Panorama Stitching
- Mathematical tool: SVD

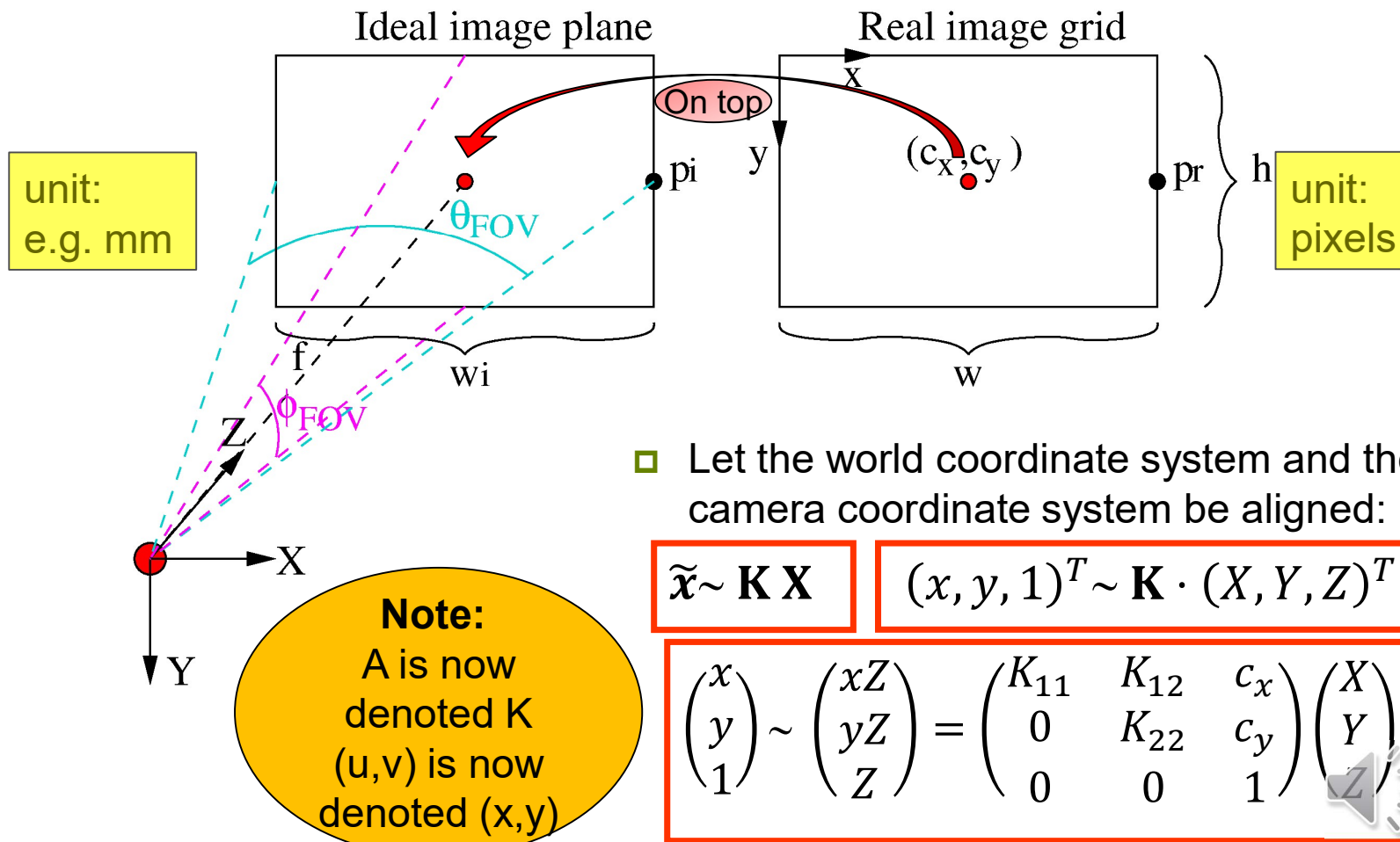
- Literature
 - “**Panorama Stitching. Supplementary Notes**” by Per-Erik Forssén
 - Last in ... “Short about camera geometry and camera calibration” by Maria Magnusson
 - Parts of ... “Introduction to Representations and Estimation in Geometry” (IREG) by Klas Nordberg
 - Parts of ... “Mathematical Toolbox for Studies in Visual Computation at Linköping University” by Klas Nordberg



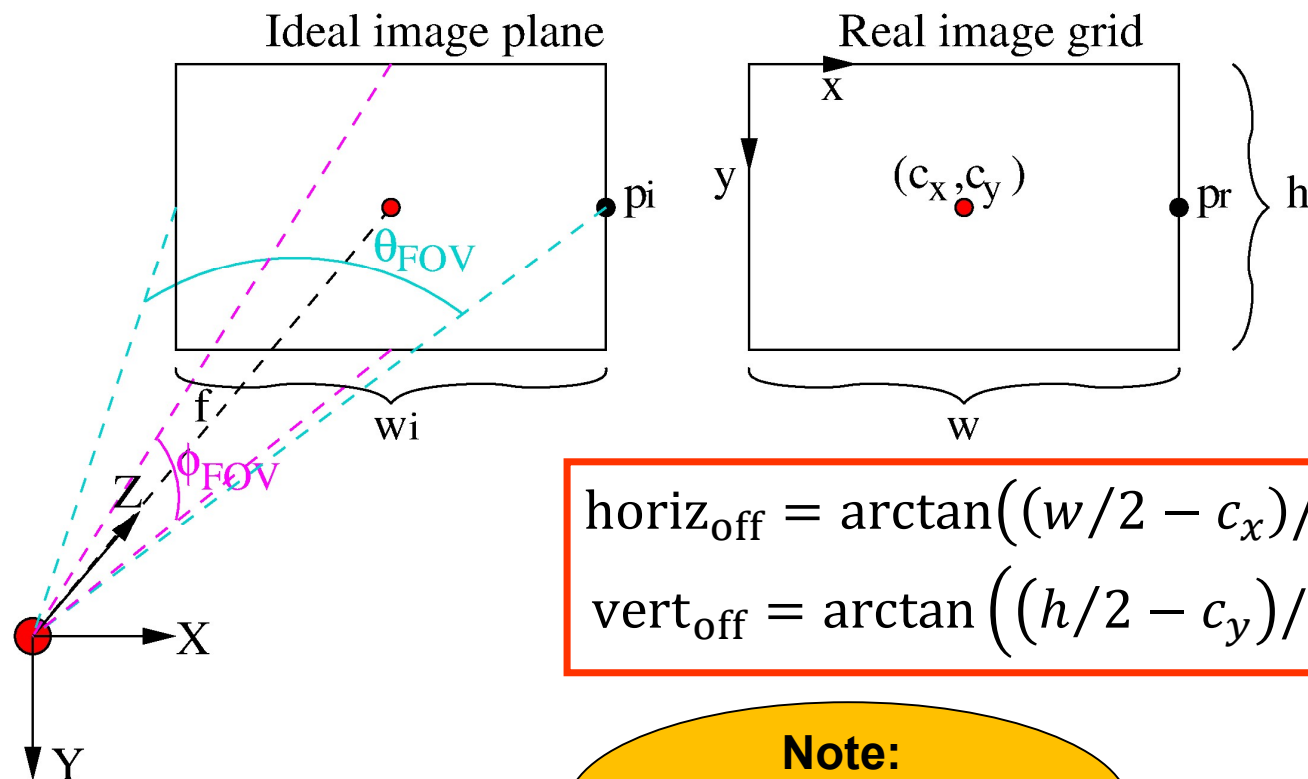
Panorama stitching



Determination of FOV and offset



Offset between optical centre and geometrical centre



$$\text{horiz}_{\text{off}} = \arctan((w/2 - c_x)/K_{11})$$

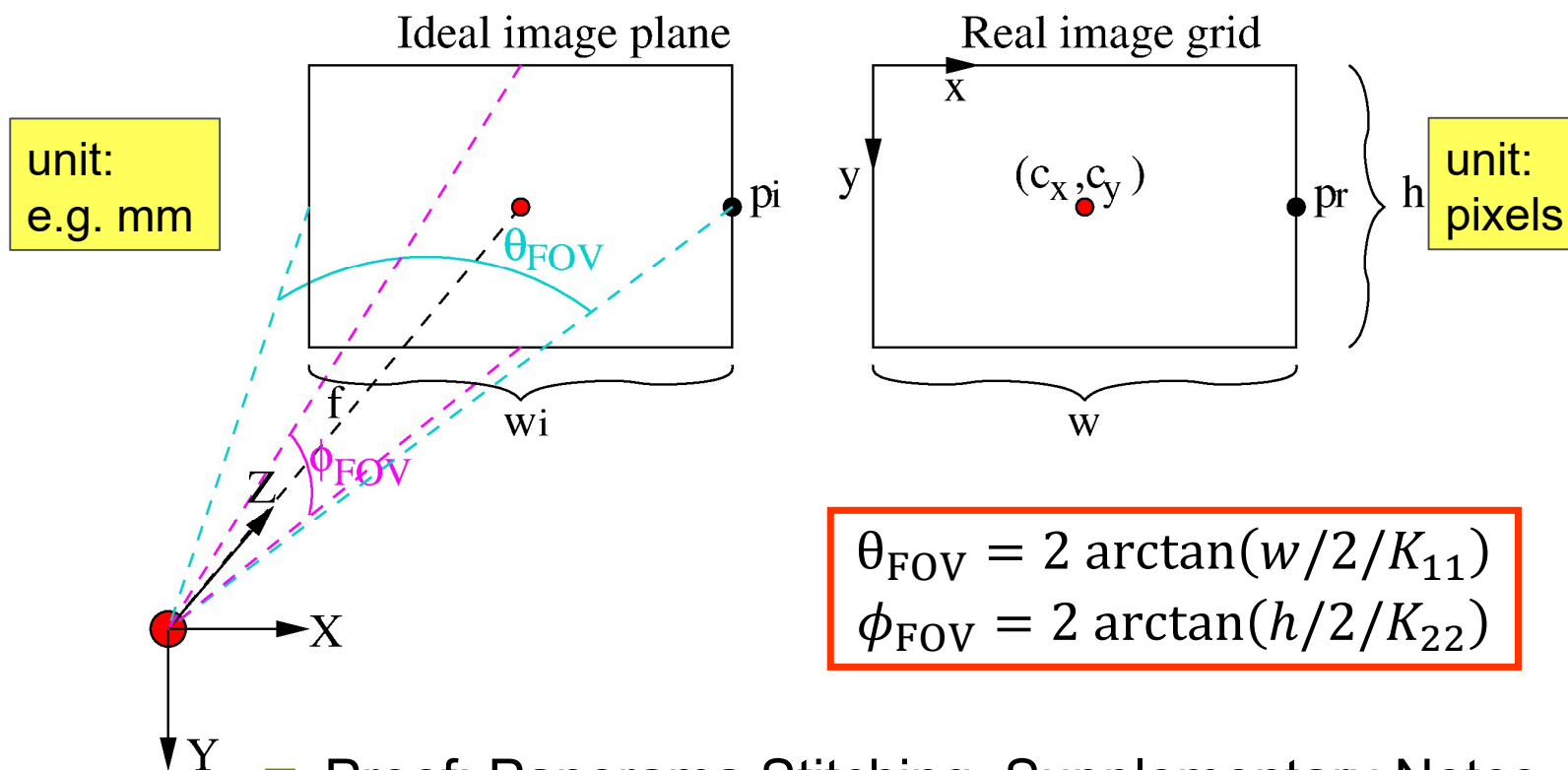
$$\text{vert}_{\text{off}} = \arctan((h/2 - c_y)/K_{22})$$

Note:

These angles are normally small.



Determination of Field Of View (FOV)



- Proof: Panorama Stitching. Supplementary Notes.
- Also in next slide.



Determination of Field Of View (FOV), proof

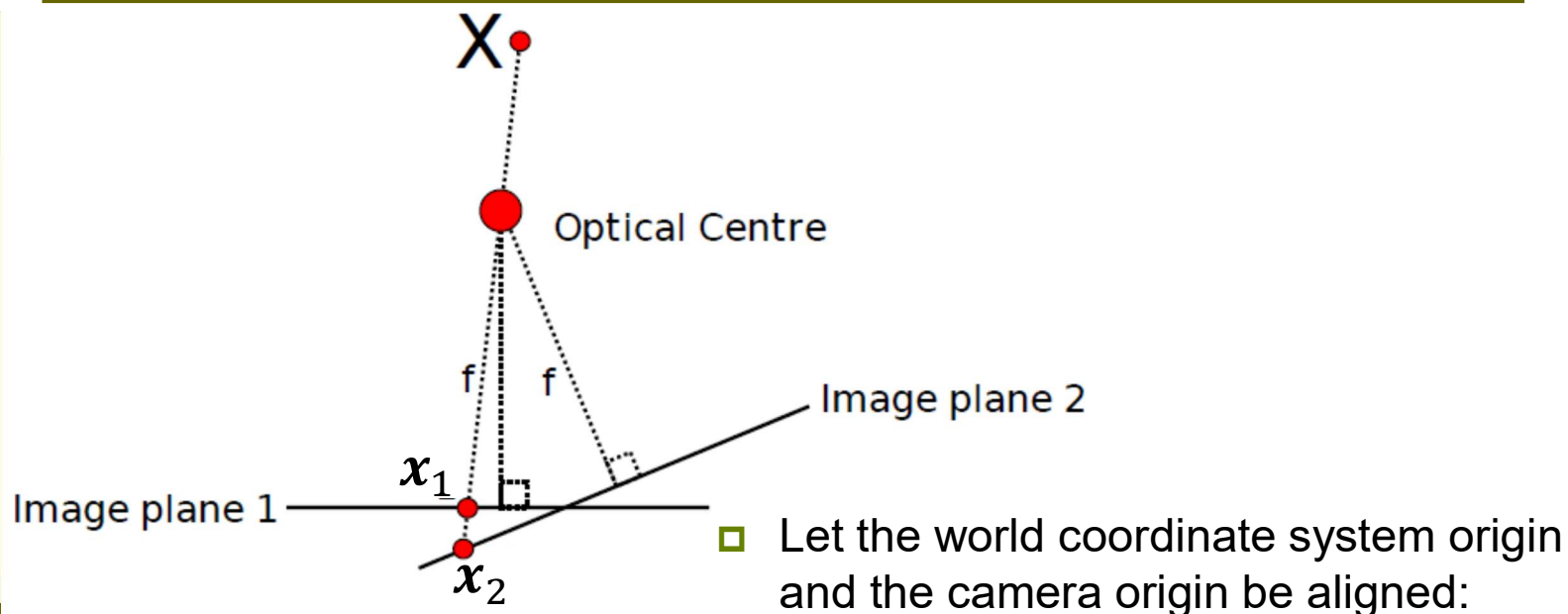
See figure 1. For simplicity, assume that $\text{horiz}_{\text{off}} = \text{vert}_{\text{off}} = 0$. Then the points $p_i = (w_i/2, 0, f)^T$ and $p_r = (c_x + w/2, c_y, 1)^T$. Therefore

$$Z \begin{pmatrix} c_x + w/2 \\ c_y \\ 1 \end{pmatrix} = \begin{pmatrix} K_{11} & K_{12} & c_x \\ 0 & K_{22} & c_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} w_i/2 \\ 0 \\ f \end{pmatrix}$$

The first row gives $Z(c_x + w/2) = K_{11}w_i/2 + c_x f$ and the third row gives $Z = f$. Therefore $f(c_x + w/2) = K_{11}w_i/2 + c_x f$, which gives $w_i = wf/K_{11}$. Finally, $\theta_{\text{FOV}} = 2 \arctan(w_i/(2f)) = 2 \arctan(w/(2K_{11}))$. Similarly, $\phi_{\text{FOV}} = 2 \arctan(h/(2K_{22}))$.



Rotational Homographies



$$\mathbf{x} = (x, y)^T$$

$$\tilde{\mathbf{x}} \sim \mathbf{K} \mathbf{R} \mathbf{X}$$

$$(x, y, 1)^T \sim \mathbf{K} \mathbf{R} \cdot (X, Y, Z)^T$$

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \sim \begin{pmatrix} K_{11} & K_{12} & c_x \\ 0 & K_{22} & c_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

Rotational Homographies

- The projections of a point \mathbf{X} in the world to the points \mathbf{x}_1 and \mathbf{x}_2 in the two images:

$$\tilde{\mathbf{x}}_1 \sim \mathbf{K} \mathbf{R}_1 \mathbf{X}$$

$$\tilde{\mathbf{x}}_2 \sim \mathbf{K} \mathbf{R}_2 \mathbf{X}$$

- Assume the existence of a homography \mathbf{H}_{21} that maps points from image2 to image1:

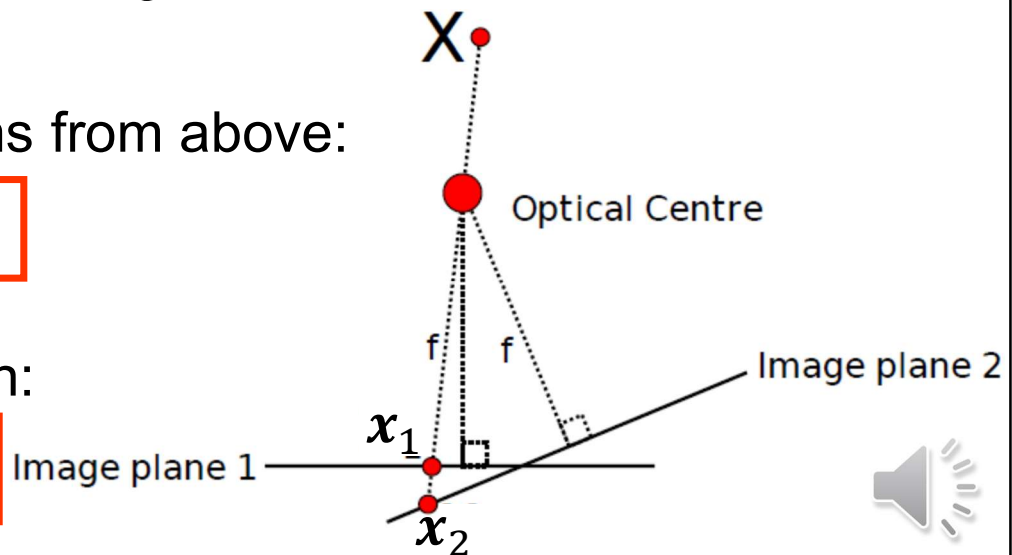
$$\tilde{\mathbf{x}}_1 \sim \mathbf{H}_{21} \tilde{\mathbf{x}}_2$$

- Insert the expressions from above:

$$\mathbf{K} \mathbf{R}_1 \mathbf{X} \sim \mathbf{H}_{21} \mathbf{K} \mathbf{R}_2 \mathbf{X}$$

- This is satisfied when:

$$\mathbf{H}_{21} = \mathbf{K} \mathbf{R}_1 \mathbf{R}_2^T \mathbf{K}^{-1}$$



Panorama stitching

- ❑ In panorama stitching, we have a set of images that share a common camera centre (origin), i.e. the images are all taken **from the same view-point** but in different directions.
- ❑ Given that the objects in the images are far away, the camera centres do not have to be exactly at the same point.
- ❑ Each image can be transformed into any other by a homography.



Panorama stitching

- By applying the homography H_{21} to image2 (which is taken by camera2), it can be **stitched** onto image1 (which is taken by camera1).
- By applying the homography H_{31} to image3 (which is taken by camera3), it can be **stitched** onto image1 (which is taken by camera1).
- If both image2 and image3 are stitched onto image1, image1 works as a **reference image**.
- It is possible to stitch a whole set of images onto one reference image.



Panorama stitching, example

- Two images taken from approximately the same view-point:

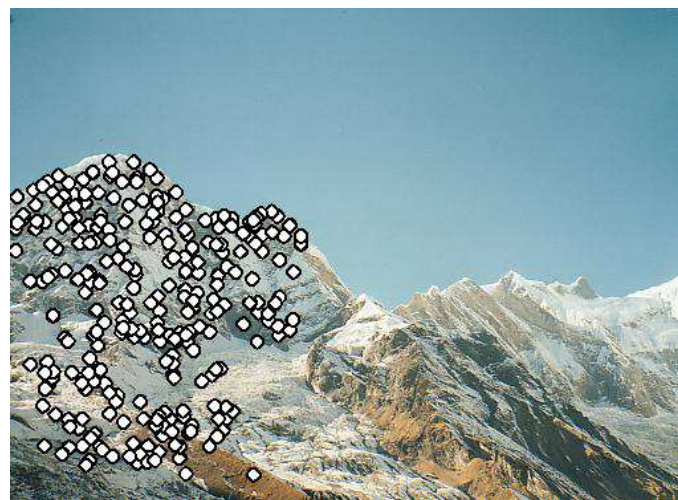
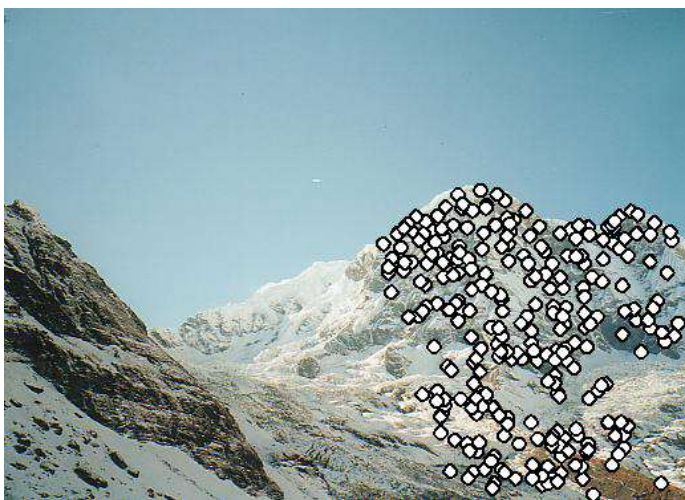


Images from: *Automatic Panoramic Image Stitching using Invariant Features*,
IJCV 2007, Matthew Brown



Panorama stitching, example

- Mark a set of corresponding points:



- From these points: Estimate a homography \mathbf{H} that relates the 2 images.



Panorama stitching, example

- The right image stitched onto the left image:



Practical issues

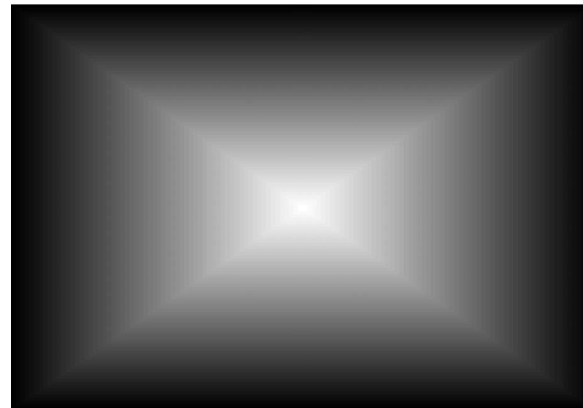
- The pixel values in overlapping regions may differ even if the geometric transformation is correct
 - Vignetting effects
 - Interpolation effects
 - Exposure time or illumination may be different in two images
 - Moving objects in the scene
- At each pixel:
 - Alternative 1: Take the value from only one of the two images
 - Alternative 2: Blend



Blending weight

- For example, use a weight that is smaller at the edges of the image and larger at the center:

This is the weight image before the homography transformation

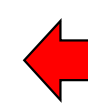


- Pano1 and Pano2 are the two images transformed to the reference grid.
- alpha1 and alpha2 are the weight image transformed to the reference grid.
- Normalized weighting:

$$Pano = \frac{\alpha1 \cdot Pano1 + \alpha2 \cdot Pano2}{\alpha1 + \alpha2}$$



Blending



With
blending

Without
blending



Practical issues

- ❑ To assure a homography between any pair of images, we must have a pin-hole camera
 - No significant amount of lens distortion is allowed
 - Alternatively: lens distortion can be estimated and compensated for before the stitching
- ❑ In the panorama computer exercise, we use the following radial distortion model,

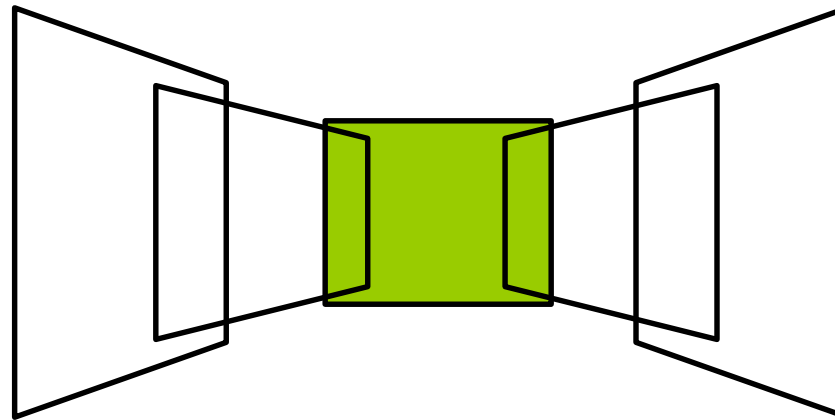
$$r_{\text{out}} = \frac{\arctan(r_{\text{in}} \cdot \gamma)}{\gamma}$$

- where the image is described in polar coordinates, (r, θ) .
- You will manually estimate γ by undistorting the images using several different γ -values. γ is good when straight lines in the world give straight lines in the image.
- γ is small, e.g. $\gamma=0.001$



Practical issues

- If the view direction between the reference image and the stitched image is very different, the 'resolution in' and the 'size of' the two images will vary a lot.
- Ex) 4 images stitched to a reference image (green):



- An attempt to stitch an image from 90° view direction onto an image from 0° view direction will result in infinity size of the stitched image.
- Solution: Map the images onto a cylinder or sphere and stitch them there instead.



Mapping a point $(X,Y,Z)^T$ to the unit sphere

- A point $(X,Y,Z)^T$ is projected to the normalized image plane $(x_n, y_n, 1)^T$ and then transformed to a point $(x, y, 1)^T$ on the real image grid as:

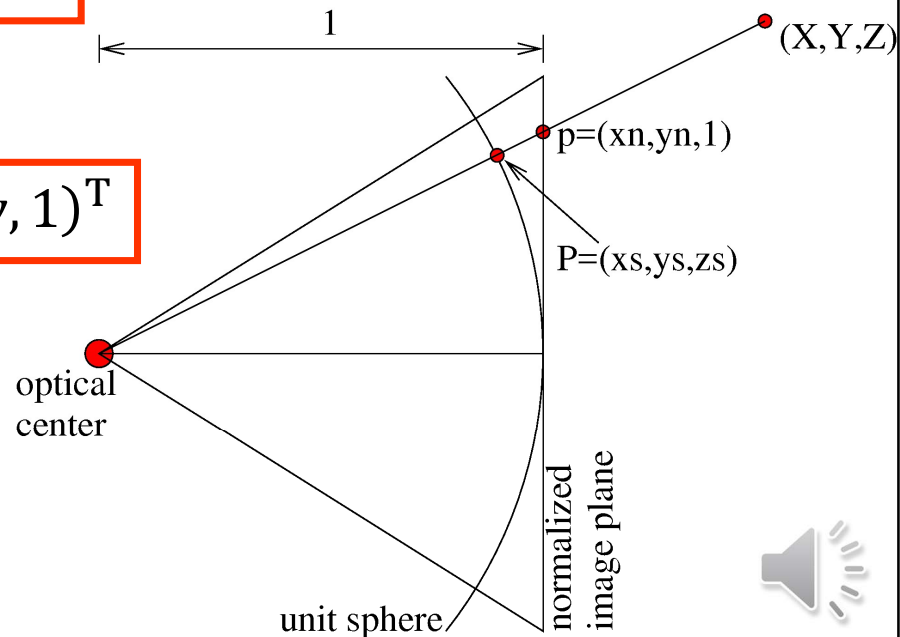
$$(x, y, 1)^T = \mathbf{K} \cdot (x_n, y_n, 1)^T$$

- Consequently:

$$(x_n, y_n, 1)^T = \mathbf{K}^{-1} \cdot (x, y, 1)^T$$

- Finally, a simple geometrical consideration gives:

$$(x_s, y_s, z_s)^T = \dots$$



The Orthogonal Procrustes problem (OPP)

- If two corresponding sets of 3D points from two images are mapped to the unit sphere, it is possible to determine the rotation between them using OPP.
- Consider two sets of 3D points $\mathbf{X} = (\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_N)$ and $\mathbf{Y} = (\mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_N)$ that are related as: $\mathbf{X}_k = \mathbf{R}\mathbf{Y}_k + \epsilon_k$,
- where \mathbf{R} is an orthogonal matrix and ϵ_k is an additive Gaussian noise term.
- It can be shown (see e.g. Panorama Stitching. Supplementary Notes) that if the SVD of \mathbf{XY}^T gives: $\mathbf{XY}^T = \mathbf{U}\mathbf{D}\mathbf{V}^T$
Then: $\mathbf{R} = \mathbf{U}\mathbf{V}^T$



Axis-Angle Representation

- When \mathbf{R} is determined, it is possible to find the rotation axis $\hat{\mathbf{n}}$ and the rotation angle $\alpha \in [0, \pi[$
- $\hat{\mathbf{n}}$ is an eigenvector of \mathbf{R} with eigenvalue 1:

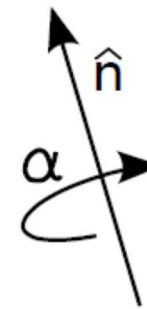
$$\hat{\mathbf{n}} = \mathbf{R} \hat{\mathbf{n}}$$

- The other two eigenvalues are:

$$e^{i\alpha}$$

$$e^{-i\alpha}$$

- Use the Matlab command $[\mathbf{V} \mathbf{D}] = \text{eig}(\mathbf{R})$;
- The eigenvectors are in \mathbf{V} and the eigenvalues are in \mathbf{D} .
- *More information on this can be found in e.g. Panorama Stitching. Supplementary...*



Resampling to Spherical Coordinates

- Now we have computed different \mathbf{R} matrices for each image to be stitched. They can then be resampled to spherical coordinates:

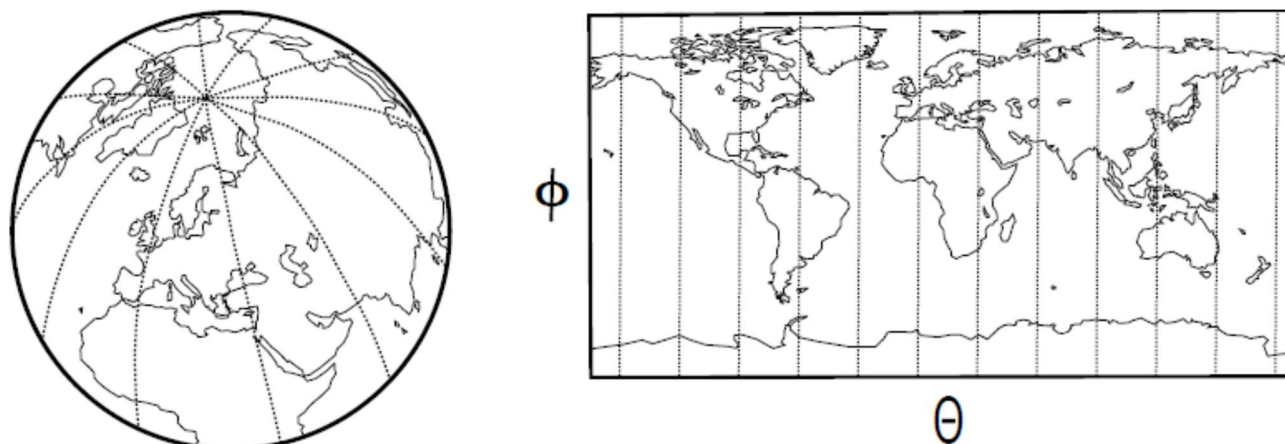


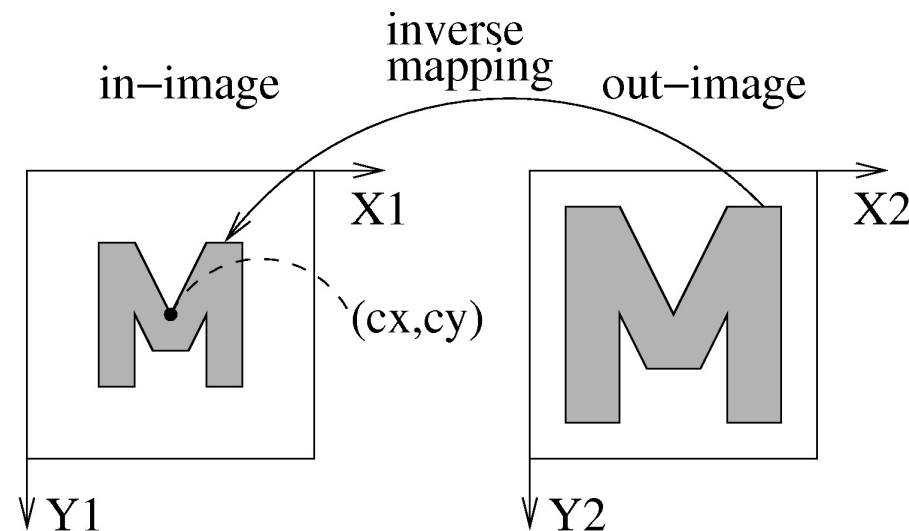
Fig. Illustration of spherical coordinates. Left: A world map painted on a sphere. Right: The same map in longitude-latitude space.



Resampling and interpolation from in-image Im1 to out-image Im2

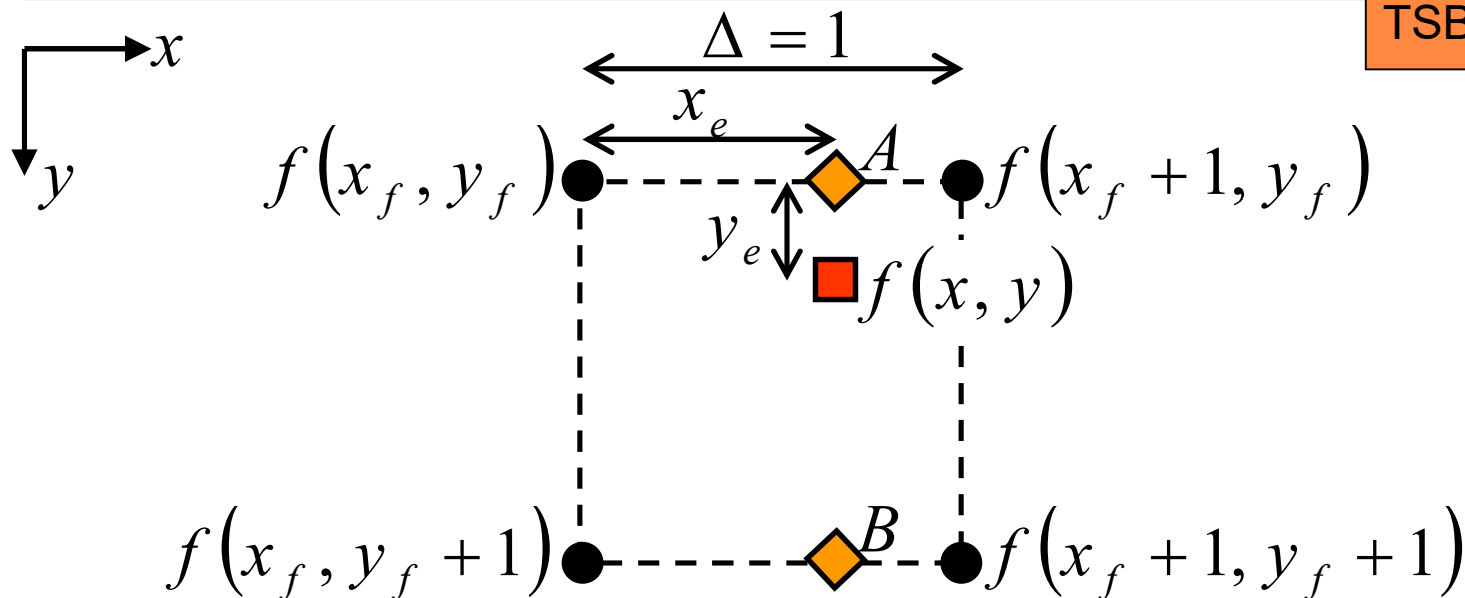
Repetition
from
TSBB08,
TSBB31

- Suppose $(X_2, Y_2)^T = \mathbf{T} \cdot (X_1, Y_1)^T$
- For all points (X_2, Y_2) in the out-image:
 - Perform an inverse mapping $\text{inv}(\mathbf{T})$ to (X_1, Y_1) in the in-image.
 - Perform interpolation, e.g. bilinear interpolation, in the in-image, obtain a value.
 - Put the value at position (X_2, Y_2) in the out-image



Repetition
from
TSBB08,
TSBB31

Bilinear interpolation



$$\begin{cases} A = f(x_f, y_f) \cdot (1 - x_e) + f(x_f + 1, y_f) \cdot x_e \\ B = f(x_f, y_f + 1) \cdot (1 - x_e) + f(x_f + 1, y_f + 1) \cdot x_e \\ f(x, y) = A \cdot (1 - y_e) + B \cdot y_e \end{cases}$$



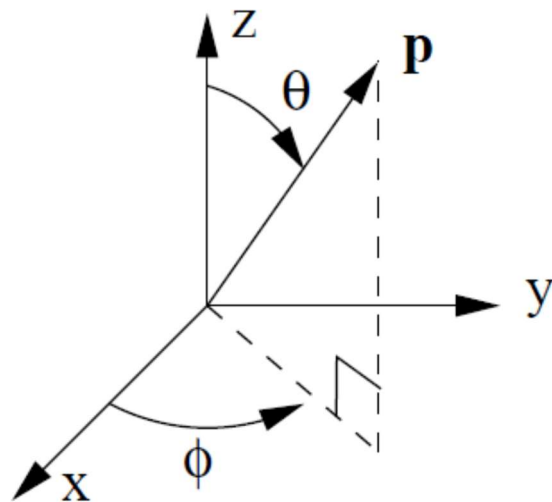
Spherical coordinate system: standard form

- Disadvantage: Singularities at $\theta = \{0, \pi\}$

Image
here

$$\mathbf{p} = (x, y, z)^T$$

$$r^2 = x^2 + y^2 + z^2$$



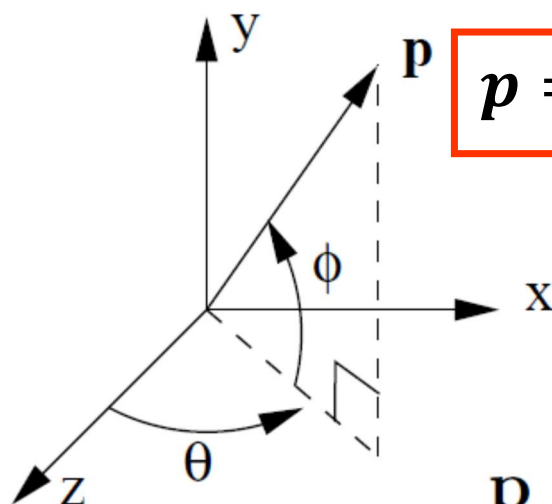
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r \sin \theta \cos \phi \\ r \sin \theta \sin \phi \\ r \cos \theta \end{pmatrix}$$



Spherical coordinate system: longitude-latitude form

□ Singularities at $\phi = \left\{-\frac{\pi}{2}, \frac{\pi}{2}\right\}$

i.e. above and below the camera, north and south pole.



$$\mathbf{p} = (x, y, z)^T$$

$$r^2 = x^2 + y^2 + z^2$$

$$\mathbf{p} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r \cos \phi \sin \theta \\ r \sin \phi \\ r \cos \phi \cos \theta \end{pmatrix}$$

Image
here



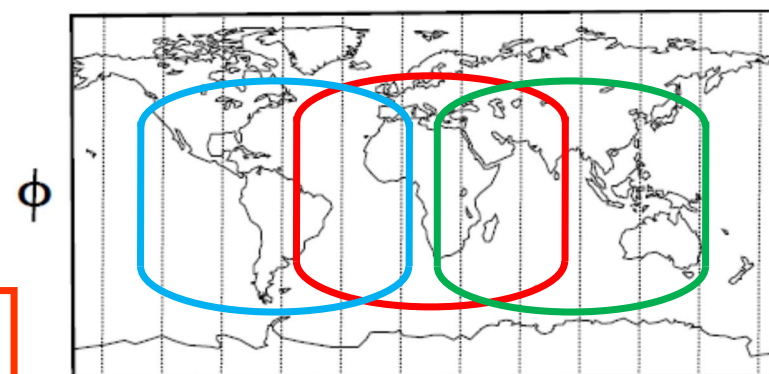
Recipe

- For all points $\mathbf{p}(\phi, \theta)$ in the output image:

- Transform it with

$$(sx, sy, s)^T = \tilde{\mathbf{x}} \sim \mathbf{K} \mathbf{R} \mathbf{p}(\phi, \theta)$$

- Receive the position (x, y) in the input image.
 - Perform bilinear interpolation in the input image, obtain a value.
 - Put the value at position (ϕ, θ) in the output image



θ Output image

Note that, normally, there are several input images, e.g. three. The reference image (**red**) have the rotation $\mathbf{R}=\mathbf{I}$, the identity matrix. The other images (**blue** and **green**) have rotations $\mathbf{R}=\mathbf{R}_1$ and $\mathbf{R}=\mathbf{R}_2$, relative to the reference image.



Singular value decomposition (SVD)

Theorem:

- For **any** $N \times M$ real valued matrix \mathbf{X} ,
 - we can find an $N \times N$ orthonormal matrix \mathbf{U}
 - we can find an $M \times M$ orthonormal matrix \mathbf{V}
 - we can find an $N \times M$ real diagonal matrix \mathbf{S}
 - such that:

$$\mathbf{X} = \mathbf{U}\mathbf{S}\mathbf{V}^T$$

- This is the **singular value decomposition (SVD)** of \mathbf{X}



Singular value decomposition (SVD)

- **S** is $N \times M$ diagonal (non-zero values only in the diagonal)
- The diagonal elements of **S**, $\sigma_1, \dots, \sigma_P$, are **real** and **non-negative** (with $P = \min(N, M)$)
- The diagonal elements of **S** are the **singular values** of **X**
- The singular values are usually ordered such that $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_P$



Singular value decomposition (SVD)

- For singular value σ_k , the corresponding columns \mathbf{u}_k and \mathbf{v}_k of \mathbf{U} and \mathbf{V} are the **left and right singular vectors** of \mathbf{X} , respectively.

- Notice that

$$\begin{aligned}\mathbf{X}\mathbf{v}_k &= \sigma_k \mathbf{u}_k \\ \mathbf{X}^T \mathbf{u}_k &= \sigma_k \mathbf{v}_k\end{aligned}$$

- Remember that

$$\mathbf{X} = \mathbf{U}\mathbf{S}\mathbf{V}^T$$



Singular value decomposition (SVD)

- In the case of a non-square matrix **X** there will be some left (or right) singular vectors that neither have a corresponding singular value σ_k nor a right (or a left) singular vector.
- In this case they are simply said to have singular value 0 since, for example

$$\mathbf{X} \mathbf{v}_k = \mathbf{0} \quad (k > P, \text{ and } N < M)$$



Solution of a homogeneous system of equations using SVD

- Regard the following homogeneous system of equations:

$$\mathbf{X}\mathbf{b} = 0$$

- Perform SVD and rewrite:

$$\begin{aligned}\mathbf{U}\mathbf{S}\mathbf{V}^T\mathbf{b} &= 0 \\ \mathbf{U}^T\mathbf{U}\mathbf{S}\mathbf{V}^T\mathbf{b} &= 0 \\ \mathbf{S}\mathbf{V}^T\mathbf{b} &= 0\end{aligned}$$

- Solution:

$$\mathbf{b} = \mu\mathbf{v}_n$$

- where μ is a scale factor and \mathbf{v}_n is the last column of \mathbf{V}

- Matlab code:

```
[U,S,V] = svd(X);  
b = V(:,n);
```



Example: Solve $\mathbf{X}\mathbf{b} = 0$

- Suppose that \mathbf{X} is a 4×3 -matrix. Since $\mathbf{X} = \mathbf{U}\mathbf{S}\mathbf{V}^T$, we have:

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \\ x_{41} & x_{42} & x_{43} \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ u_{21} & u_{22} & u_{23} & u_{24} \\ u_{31} & u_{32} & u_{33} & u_{34} \\ u_{41} & u_{42} & u_{43} & u_{44} \end{pmatrix} \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_{11} & v_{21} & v_{31} \\ v_{12} & v_{22} & v_{32} \\ v_{13} & v_{23} & v_{33} \end{pmatrix}$$

- Previous slide gave: $\mathbf{S}\mathbf{V}^T\mathbf{b} = 0$

- Solution: $\mathbf{b} = \mu\mathbf{v}_3 = \mu(v_{13}, v_{23}, v_{33})^T$ Insert:

$$\begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_{11} & v_{21} & v_{31} \\ v_{12} & v_{22} & v_{32} \\ v_{13} & v_{23} & v_{33} \end{pmatrix} \mu \begin{pmatrix} v_{13} \\ v_{23} \\ v_{33} \end{pmatrix} = \mu \begin{pmatrix} 0 \\ 0 \\ \sigma_3 \end{pmatrix}$$

- The smaller σ_3 , the better solution and $\sigma_3 = 0$ solves $\mathbf{X}\mathbf{b} = 0$ perfectly.

