TSBB09 Image Sensors, Projective Geometry, Lecture 4A

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Literature: Parts of ...
”Introduction to Representations and Estimation in Geometry”, by Klas Nordberg

A vector space

- A vector space *V* consists of a set of vectors
  - Two vectors can be added
  - A vector can be multiplied by a scalar
  - Both operations result again in a vector in *V*
- The dimension of *V* =
  maximal number of vectors which are linear independent
- Typically: there exists one or another basis
- Orthogonality between two vectors defined if we have a scalar product
- Linear mappings are well-defined

A projective space

- A projective space can be defined from V in terms of equivalence classes:
  - Two vectors **u** and **v** are **equivalent** if there exists a non-zero scalar λ such that **u**=λ**v**
    => **u** and **v** must be non-zero vectors
  - All vectors which are equivalent correspond to an element of the projective space
    (a projective element)
  - Projective equivalence is denoted **u** ~ **v**
- The projective space is (often) denoted $\mathbb{P}(V)$

Projective representation

- The *n*-dimensional vector space $\mathbb{R}^n$ can be given a projective representation by the projective space $\mathbb{P}(\mathbb{R}^{n+1})$

$$\bar{\mathbf{v}} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n$$

is represented by the projective element corresponding to

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \\ 1 \end{pmatrix} \in \mathbb{R}^{n+1}$$

Homogeneous coordinates

- The vector **v** is called the *homogeneous coordinates* of $\bar{\mathbf{v}}$
- The projective representation of $\mathbb{R}^n$ is not unique
  - The extra dimension can be inserted at arbitrary position
  - The constant value can be arbitrary (but fix and non-zero)
  - The "one-last" representation is the most common in the literature

Example

$$\bar{\mathbf{v}} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \Rightarrow \mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \sim \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix} \sim \begin{pmatrix} \frac{1}{2} \\ 1 \\ \frac{1}{2} \end{pmatrix}$$

All these vectors in $\mathbb{R}^3$ represent the same projective element

Normalization

- Given a <u>*non-zero*</u> vector $\boldsymbol{u} \in \mathbb{R}^{n+1}$ we can scale it so that the last element = 1

  This is denoted *normalization*
- The other elements in the normalized homogeneous vector are the vector in $\mathbb{R}^n$ that **u** represents
- This makes it possible to know which vector in $\mathbb{R}^n$ a specific projective element in $\mathbb{P}(\mathbb{R}^{n+1})$ represents

2D Coordinate transformations

- A 2D point **y** is transformed to **y'** such that the corresponding Cartesian 2D coordinates are related as

$$\begin{pmatrix} y'_1 \\ y'_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

Referred to as an ***affine transformation***
Includes translation, scaling, rotation, skewing

An ***affine transformation*** transforms parallel lines to parallel lines.

Coordinate transformations

- Translation:

$$\begin{pmatrix} y'_1 \\ y'_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

- Rotation:

$$\begin{pmatrix} y'_1 \\ y'_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

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Coordinate transformations

- In homogeneous coordinates this is

$$\mathbf{y}' = \begin{pmatrix} y'_1 \\ y'_2 \\ 1 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ 1 \end{pmatrix}$$

$$\mathbf{y}' = \begin{pmatrix} y'_1 \\ y'_2 \\ 1 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{y}$$

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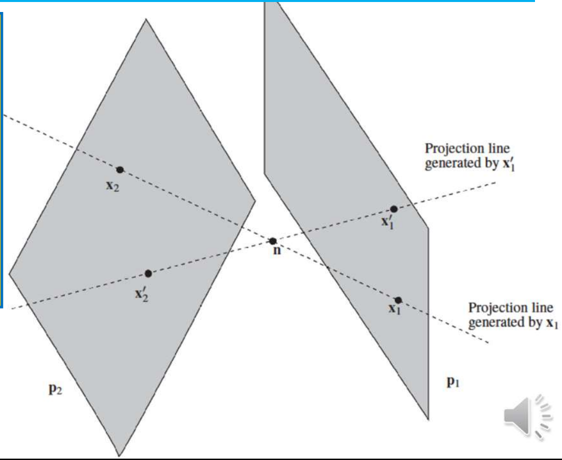
2D Homography mapping

A 2D homography maps the points $\mathbf{x}_1$ and $\mathbf{x}'_1$ in the plane $\mathbf{p}_1$ to points $\mathbf{x}_2$ and $\mathbf{x}'_2$ by projecting each point through the point **n** and finding the intersection of the projection line with the plane $\mathbf{p}_2$.



3 x 3 matrix

$$\boldsymbol{x}_2 = \boldsymbol{H}\boldsymbol{x}_1$$

Homography

- Geometrically, we define a homography as a mapping between two 2D planes in the 3D space $\mathbb{R}^3$, by projecting through a fixed point **n**.
- We assume that **n** is not included in any of the two planes $\Rightarrow$ **H** is always invertible.
- Since the matrix **H** has 9 elements but scalar multiplication does not matter, a homography has 8 degrees of freedom. Consequently, in the case of a 2D homography transformation we need at least 4 points, before and after the transformation, to determine which homography it is.

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Homography

- Describes e.g. how a pinhole-camera maps points on a plane to the image plane.
- A homography maps a line to a line.
- Parallel lines are in general **not** transformed to parallel lines.

Photo of a painting of Växjö (my hometown). Parallel lines along the frame in the painting are not parallel in the photo.