TSBB21 Image Sensors

Image Formation
Part 3

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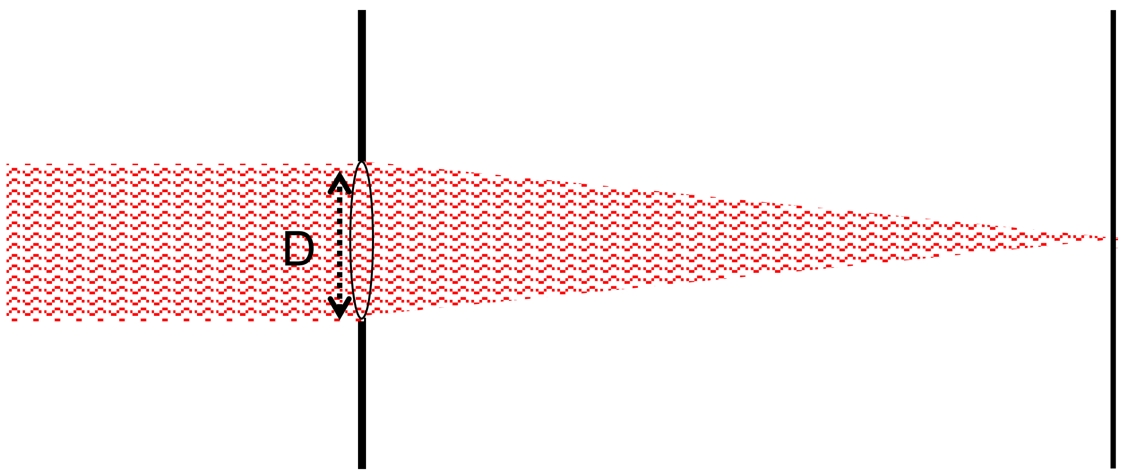
Diffraction limited systems

- Due to the wave nature of light, even when various lens effects are eliminated, light from a single 3D point cannot be focused to an arbitrarily small point if it has passed an aperture
- For coherent light:
  - Huygens' principle: treat the incoming light as a set of point light sources
  - Gives *diffraction* pattern at the image plane

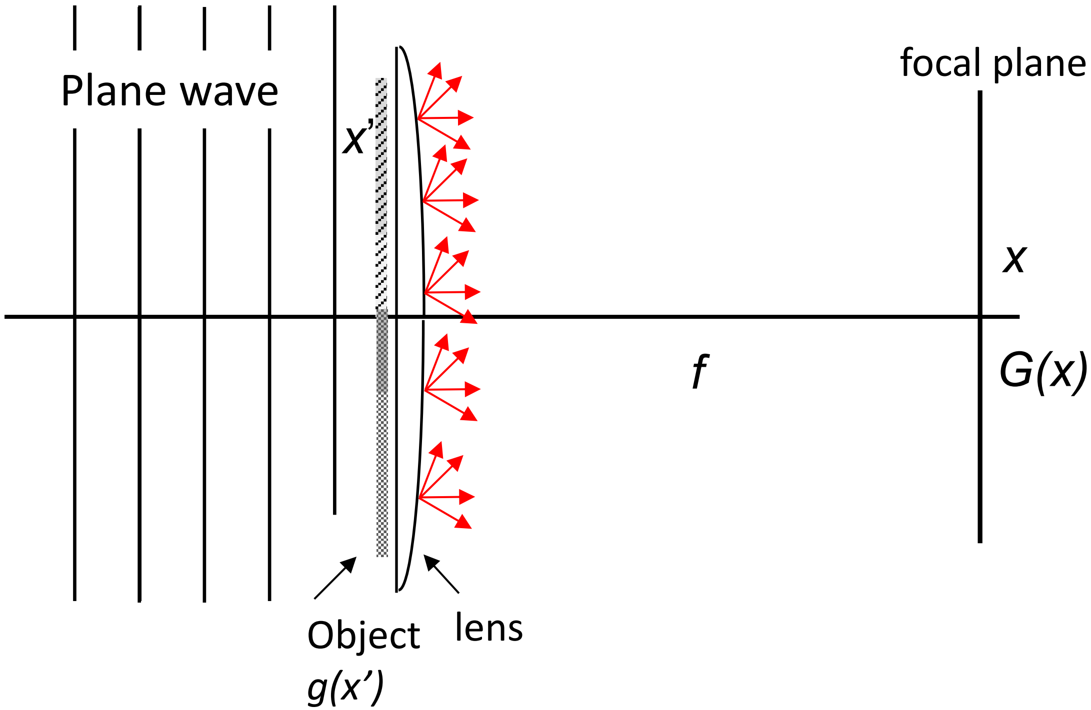
Diffraction limited systems

- Consider an ideal lens with aperture size D:



- Because of diffraction, a point source infinitely far away (a planar wave) will not be focused onto a single point in the image plane. How come?

A lens produces the Fourier transform!



- We will show that $G(x)$ is the Fouriertransform of $g(x')$ (apart from a phase factor)
- Using Huygens' wave model for light.

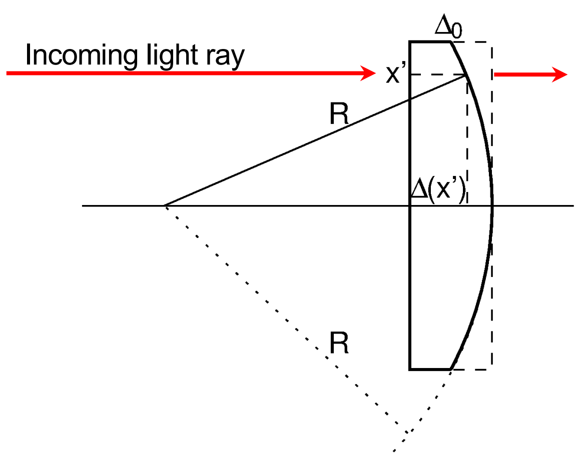
Deriving the lens transform

- Add the light contribution from each point *x′* entering the lens to each point *x* in the focal plane, taking magnitude *A* and phase ϕ into account.
- Magnitude is given by the object density *g*(x'). Phase depends on the optical path length. Compute this separably for the lens and the path from the lens to the focal plane
- Math trick: represent each light contribution by a complex number $Ae^{j\phi}$ where A is the magnitude and ϕ is the phase relative to a common reference.
- 1D-analysis (can easily be extended to 2D)

Path length through the lens

Simplifications

- The lens is plano-convex and thin
- Paraxial approximation
- Coherent light
- Inscribe the lens within a virtual rectangular box and apply Huygens' principle on the light coming out from this box.



$$\Delta(x') = \Delta_0 - \left(R - \sqrt{R^2 - x'^2}\right) \approx \Delta_0 - \frac{x'^2}{2R}$$

Where we have used x' << R (paraxial approximation)

Assume that the light travels slower by a factor *n* (*refractive index*) in the lens than in air and exits at the same height (x') since the lens is thin.

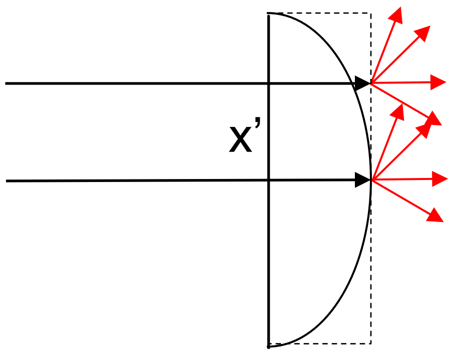
The "optical path length" travelled within the virtual box will then be

$$\delta(x') = n\Delta(x') + \left(\Delta_0 - \Delta(x')\right) = n\Delta_0 - (n-1)\frac{x'^2}{2R}$$

Applying "Lensmakers Formula" *1/f* = (*n*-1)/*R* gives:

$$\delta(x') = n\Delta_0 - \frac{x'^2}{2f}$$

The phase transform for the lens



The optical pathlength $\delta(x')$ corresponds to the phase shift:

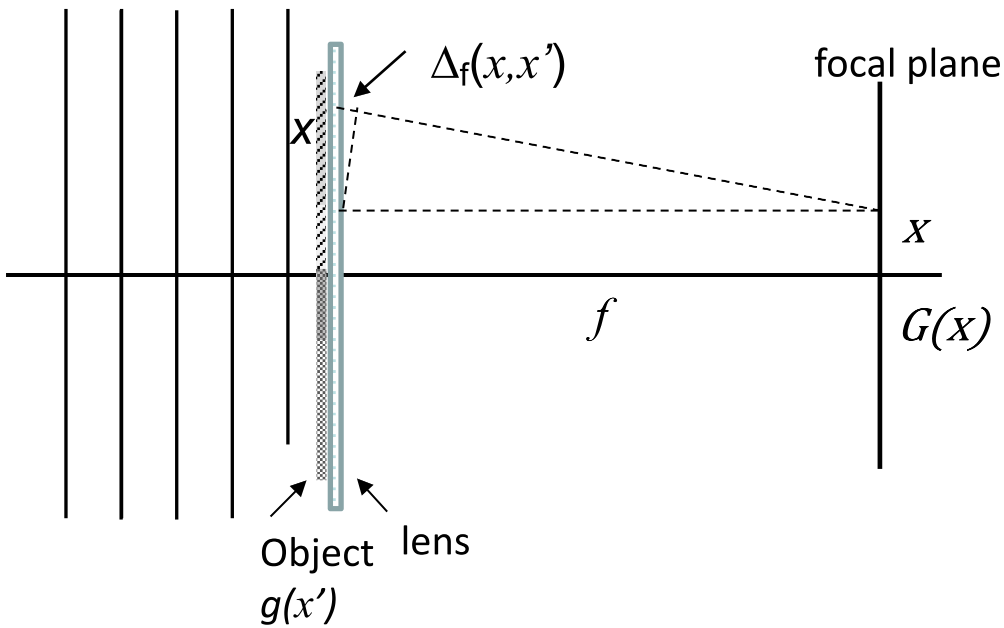
$$\Delta\phi = \frac{2\pi}{\lambda}\delta(x')$$

Inserting $\delta(x') = n\Delta_0 - \frac{x'^2}{2f}$

and disregarding the fixed delay $n\Delta_0$ gives the "phase transform":

$$T_L(x') = e^{-j\frac{k}{2f}x'^2} \quad \text{where} \quad k = \frac{2\pi}{\lambda}$$

The phase transform from lens to focal plane



$$\Delta_f(x,x') = \sqrt{(x'-x)^2 + f^2} - f \approx \frac{(x'-x)^2}{2f}$$

(again using the paraxial approximation)

Thus $T_f = e^{j\frac{k}{2f}(x'-x)^2}$

Putting it all together

$$G(x) = \int_{-\infty}^{\infty} g(x') \cdot T_L \cdot T_f dx' = \int_{-\infty}^{\infty} g(x') \cdot e^{-j\frac{k}{2f}x'^2} \cdot e^{j\frac{k}{2f}(x'-x)^2} dx' =$$

$$= e^{j\frac{k}{2f}x^2} \int_{-\infty}^{\infty} g(x') e^{-j\frac{k}{f}xx'} dx'$$

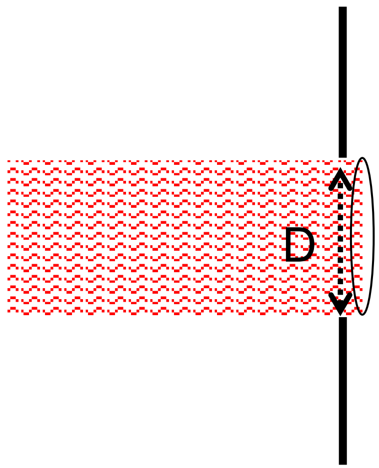
Normalise according to $u = x/(\lambda f)$

$$\boldsymbol{G}(u) = e^{j\pi\lambda f u^2} \int_{-\infty}^{\infty} g(x') e^{-j2\pi u x'} dx'$$

This is the Fourier Transform multiplied by a phase factor (of magnitude 1)!

* See the course web page how to get rid of the phase factor by moving the object further away from the lens.

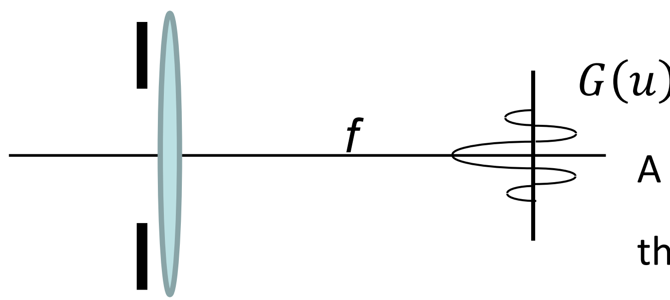
The effect of the aperture



The aperture can be viewed as an input image: $g(x') = rect(x'/D)$

The lens produces:

$$G(u) = \left(e^{j\pi\lambda f u^2}\right)\frac{\sin(\pi D u)}{\pi D u}$$



A screen at the image plane will show the (diffraction) pattern: $\left|\frac{\sin(\pi D u)}{\pi D u}\right|^2$

Any projected image will be convolved (blurred) by *G(u)*!

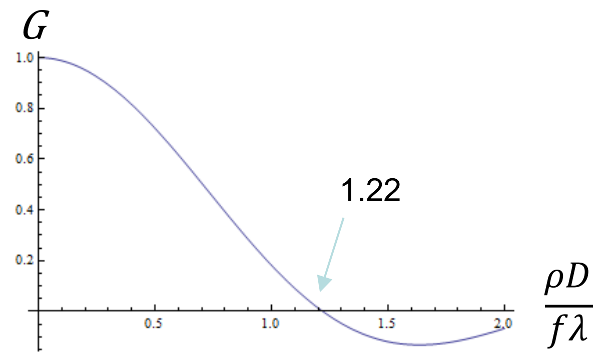
Diffraction limited systems

- $\frac{\sin(x)}{x}$ is termed the "*sinc*" function
- This phenomena generalizes to 2D:
  - The resulting wave-function $G(u, v)$ is the 2D FT of the incoming spatial amplitude $g(x', y')$
  - Example: a circular aperture of diameter *D*

(Input amplitude normalized to 1/*fλ*, $r = \sqrt{x'^2 + y'^2}$, $\rho = \sqrt{u^2 + v^2}$)

$$g(r) = \frac{1}{f\lambda} rect\left(\frac{r}{D}\right)$$

$$G(\rho) = \frac{J(\pi\rho D/f\lambda)}{\pi\rho D/f\lambda)}$$

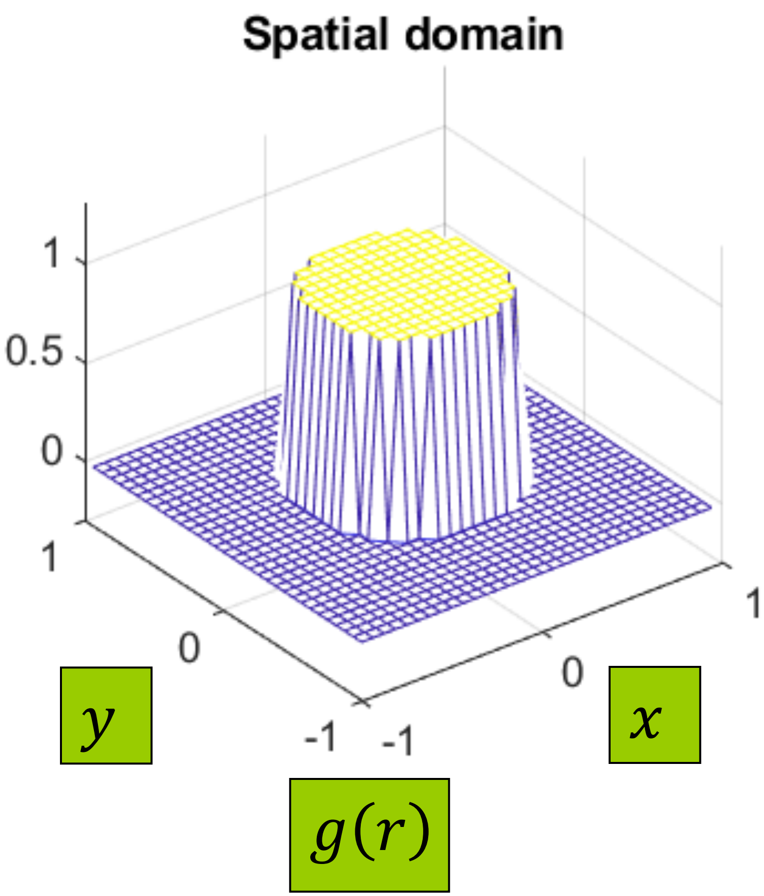


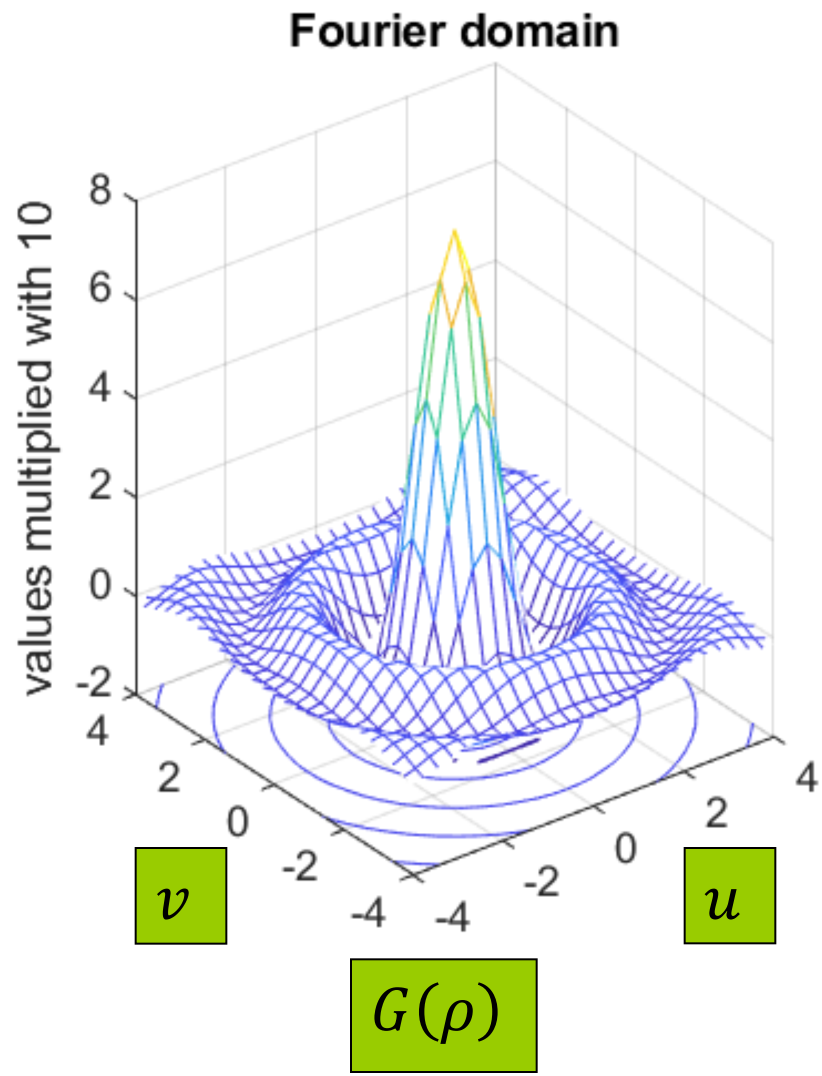
First order Bessel function

The Fourier transform of a circular disc

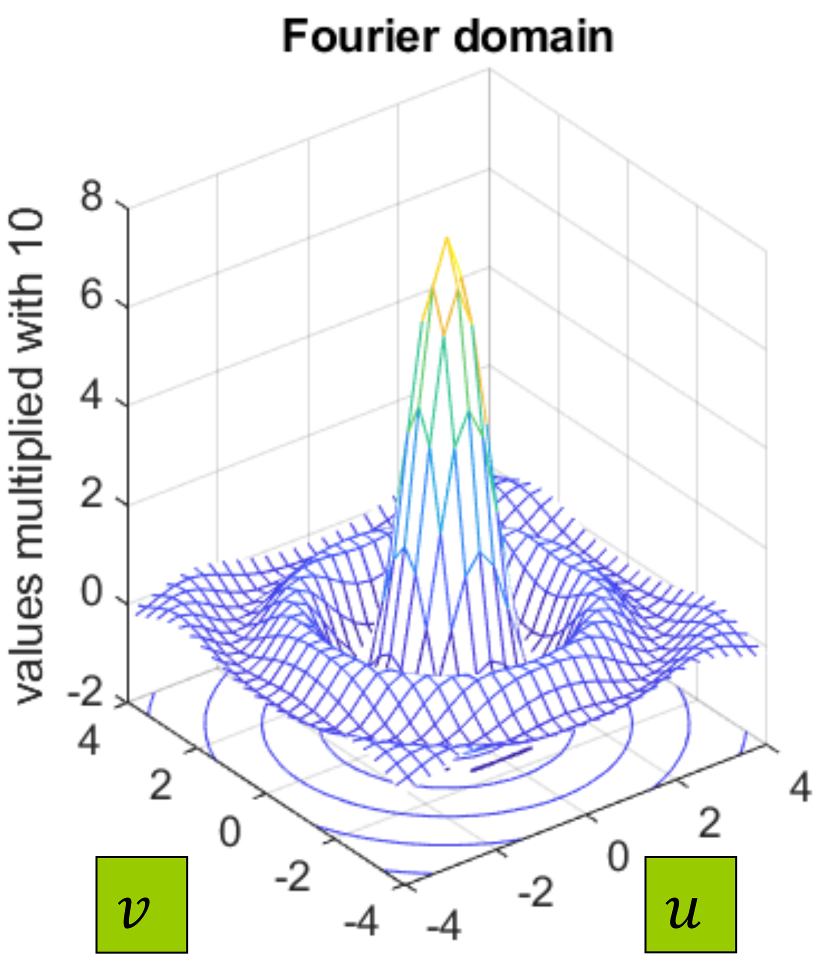
- The circular box $g(r)$ and its Fourier transform $G(\rho)$
  - $G(\rho)$ is sometimes called the *jinc* function because of its similarities with the *sinc* function.
  - To make the image visible, we need to put a screen in the image plane. The screen will only produce the intensity (the square of the magnitude), not the phase of the light (and not negative light)!
  - $G^2(\rho)$ is called the *Airy pattern*. This means that a point-source at infinite distance (planar wave front) will give rise to an Airy pattern when projected by a lens through the aperture.

The circular box and its Fourier transform



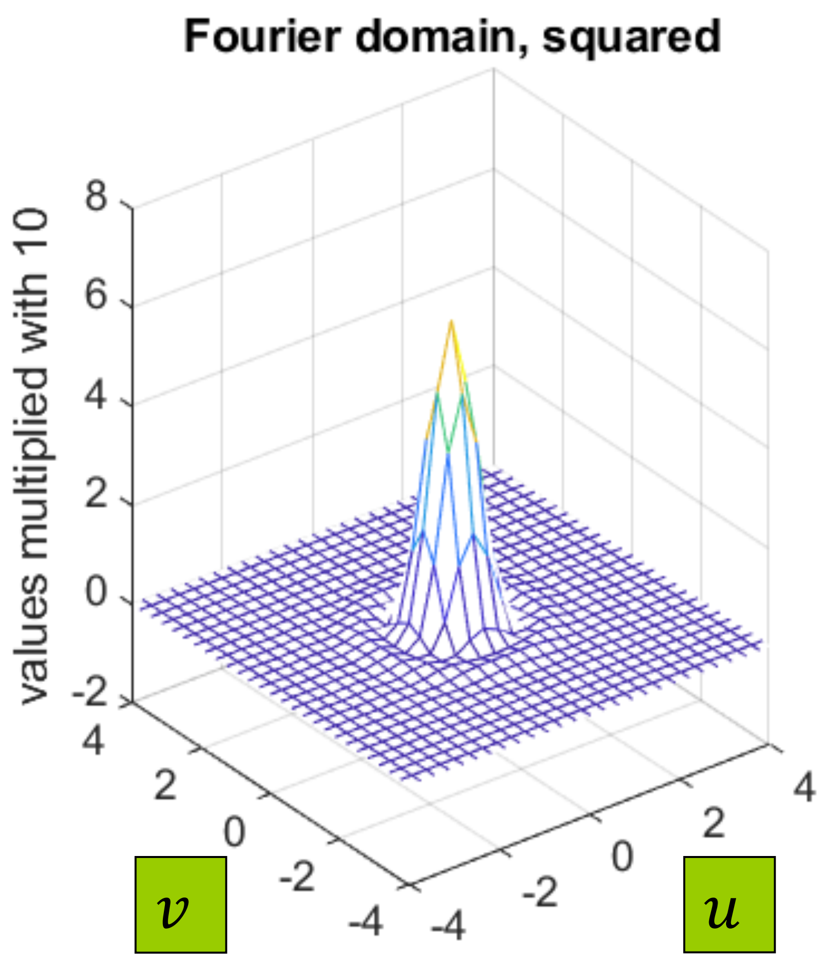


The Fourier transform and the squared Fourier transform



v u

$G(\rho)$

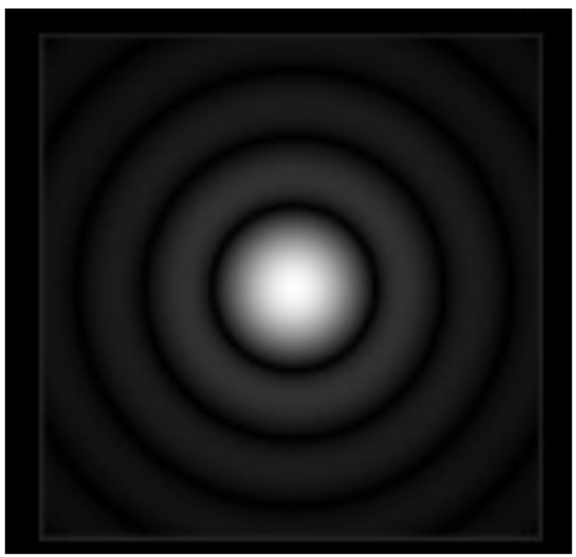


v u

$G^2(\rho)$

The Airy disk

The image of a point-source for a diffraction-limited optical system is called *Airy pattern*. The central part is called the *Airy disk*.



Airy pattern: The image of a focussed point-source becomes a diffraction pattern consisting of concentric light and dark circles.

The distance from center to first dark ring is 1.22 $f\lambda/D$.

The light intensity is given by the square of the jinc-function.

Resolution limit

- The smallest resolvable distance, Δx, between two point-sources in the image plane is given by

Distance to first zero point in Ã(x)

lens focal length

$$\Delta x \approx 1.22\,\lambda\,\frac{f_L}{D}$$

lens diameter

light wavelength

Resolution limit

Conclusions:

- The image cannot have a better resolution than Δx
- No need to measure/digitise the image with higher resolution than Δx !

- Be aware of cameras with high pixel resolution and high diffraction (e.g. small aperture)
  - Image resolution is not defined by number of pixels in the camera, rather by the diffraction limit.

Point spread function (PSF)

- **PSF** is a generalization of the point light source response
  - Diffraction: results in the Airy disk function
  - Out-of-focus blur: The out-of-focus PSF takes the shape of the camera aperture. For a circular aperture, the PSF is a disk, which is sometimes referred to as the circle of confusion.
- There are also other factors that contribute to the point spread function
  - Atmospheric turbulence
  - Optical aberrations
  - Motion
  - etc.
- The fact that the out-of-focus PSF takes the shape of the camera aperture is utilized for coded apertures, see the lecture on Specialized cameras.

Optical transfer function (OTF)

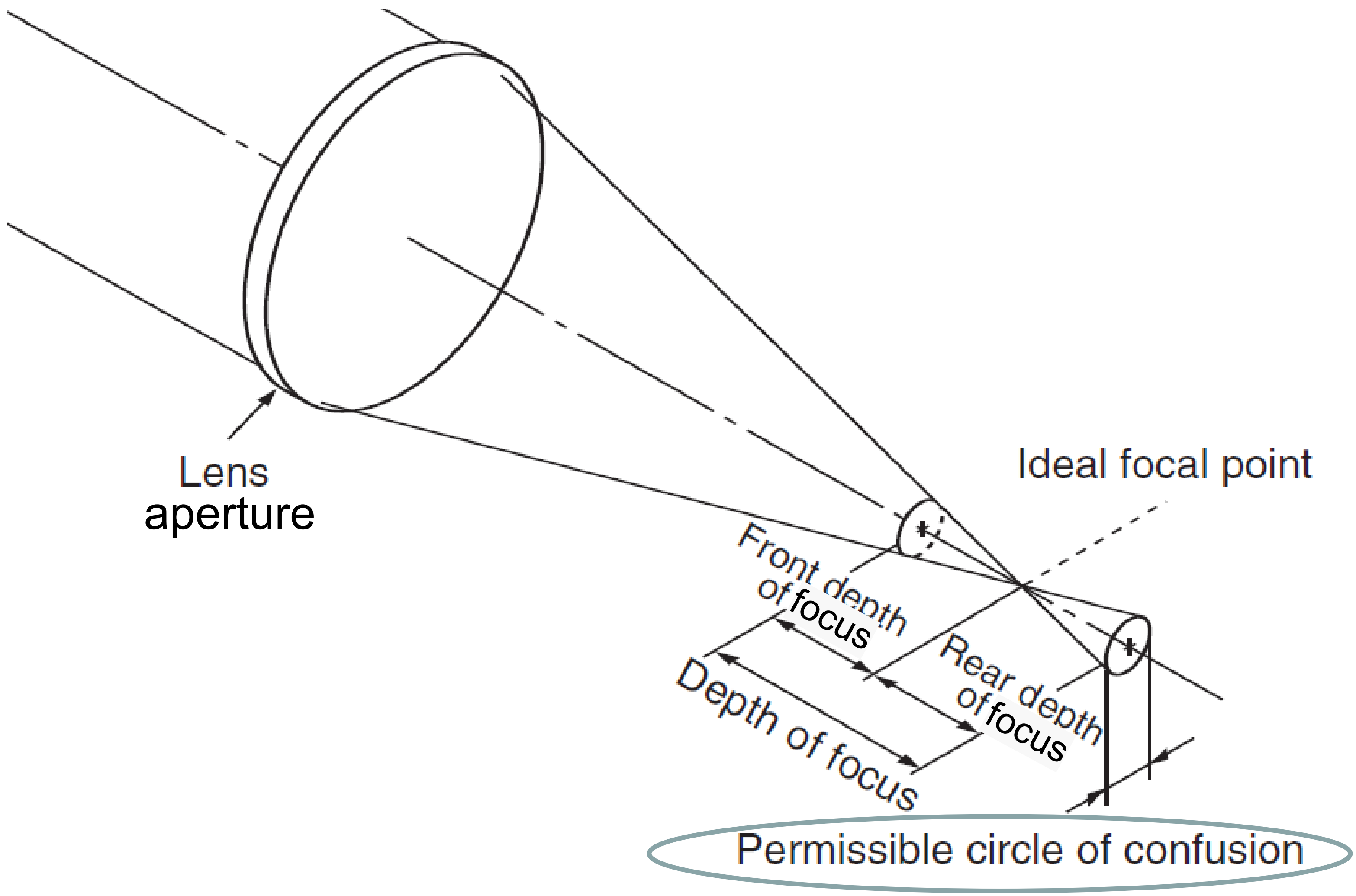
- The optical transfer function (OTF) of an optical system specifies how different spatial frequencies are handled by the system.
- OTF is the Fourier transform of the PSF
- For an ideal lens system, in focus, the OTF is the Fourier transform of the Airy disk.
- Summary and observation
  - 1) The Fourier transform of the circular box is a jinc function.
  - 2) Similarly, the Fourier transform of the jinc function is a circular box.
  - 3) The Airy disk is the jinc function multiplied with itself.
  - 2) and 3) gives that the OTF, the Fourier transform of the Airy disk, is a circular box convolved with itself!

Depth of field

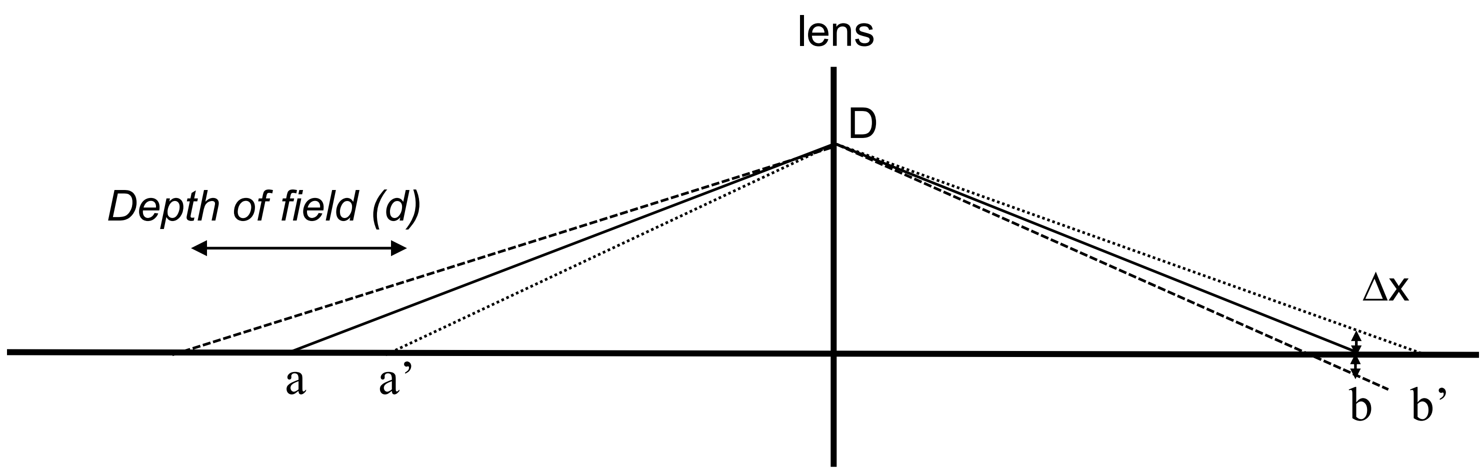
"Skärpedjup" in Swedish

- The lens gives a focused image
  - Points that are off the object plane become blurred proportional to the displacement from the object plane
- Due to the resolution limit, it makes sense to accept blur in the order of Δx
  - This blur will be there anyway due to diffraction
- *Depth of field d* is the displacement along the optical axis from the object plane that gives blur $\leq \Delta x$

Depth of field



Depth of field



$$\frac{1}{a}+\frac{1}{b}=\frac{1}{f_L}$$

- Insert a' = a - d/2 to get the "horizontal blur" (b'-b)
- Relate "horizontal blur" to vertical blur Δx

Depth of field

- For a camera where $a < \infty$, an approximation (assuming $d << a$) for d is

$$\boxed{d \approx 2\Delta x \frac{a(a - f_L)}{Df_L}}$$

a = distance from lens to object plane

f_L = lens focal length

D = lens diameter

Δx = required image plane resolution

d = depth of field

Depth of field

- For a lens where a = ∞, points that are further away than d_{min} are blurred less than Δx where

$$\boxed{d_{\min} = \frac{f_L D}{4\Delta x}}$$

The F-number

- f_L/D is the *F-number* of the lens or lens system

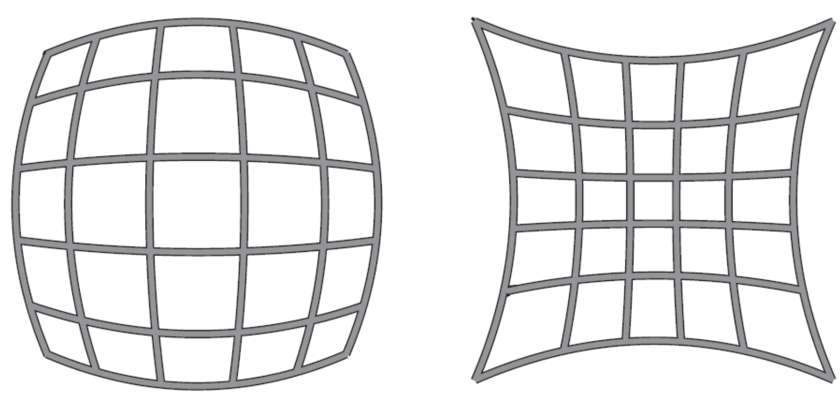
- Example
  - A typical *F* number of a camera = 8
  - Blue light = 420 nm wavelength
  - Airy disk radius $\Delta x = 1.22\ \lambda F \approx 4\ \mu\text{m}$

- For a lens with f_L = 15 mm we get
  - $d \approx 0.6$ m at $a = 1.5$ m
  - $d_{min} \approx 1.8$ m at $a = \infty$

This means that the depth of field is within a manageable range

Lens distortion

- A lens or a lens system can never map straight lines in the 3D scene exactly to straight lines in the image plane
- Depending on the lens type, a square pattern will typically appear like a *barrel* or a *pincushion*



Lens distortion

Barrel distortion

No distortion

Pincushion distortion

Radial lens distortion

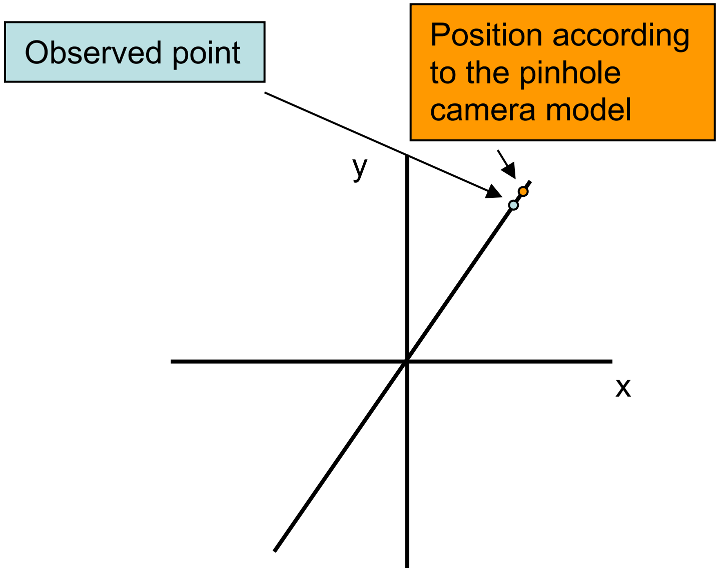
- This effect is called *lens distortion* (*geometric distortion*) and can, in the simplest case, be modeled as a *radial distortion*

  (x, y) = correct image coordinate

  $(x, y) = r\,(\cos\theta, \sin\theta)$

  (x', y') = observed image coordinate

  $(x', y') = h(r)\,(\cos\theta, \sin\theta)$



- The observed positions of points in the image are displaced in the radial direction relative the image center as described by the pinhole camera model.

Radial lens distortion

- *h* is approximately a linear function with some non-linear deviation, e.g.

$$h(r) = \frac{r}{1+\kappa\, r^2}$$

$$h(r) = r + \kappa_2 r^2 + \kappa_3 r^3 + \ldots$$

$$h(r) = \frac{1}{\omega} \tan^{-1}(2r \tan \frac{\omega}{2})$$

The deviation from a linear function usually grows with *r*

- Once modeled, we can compensate for the distortion

Vignetting

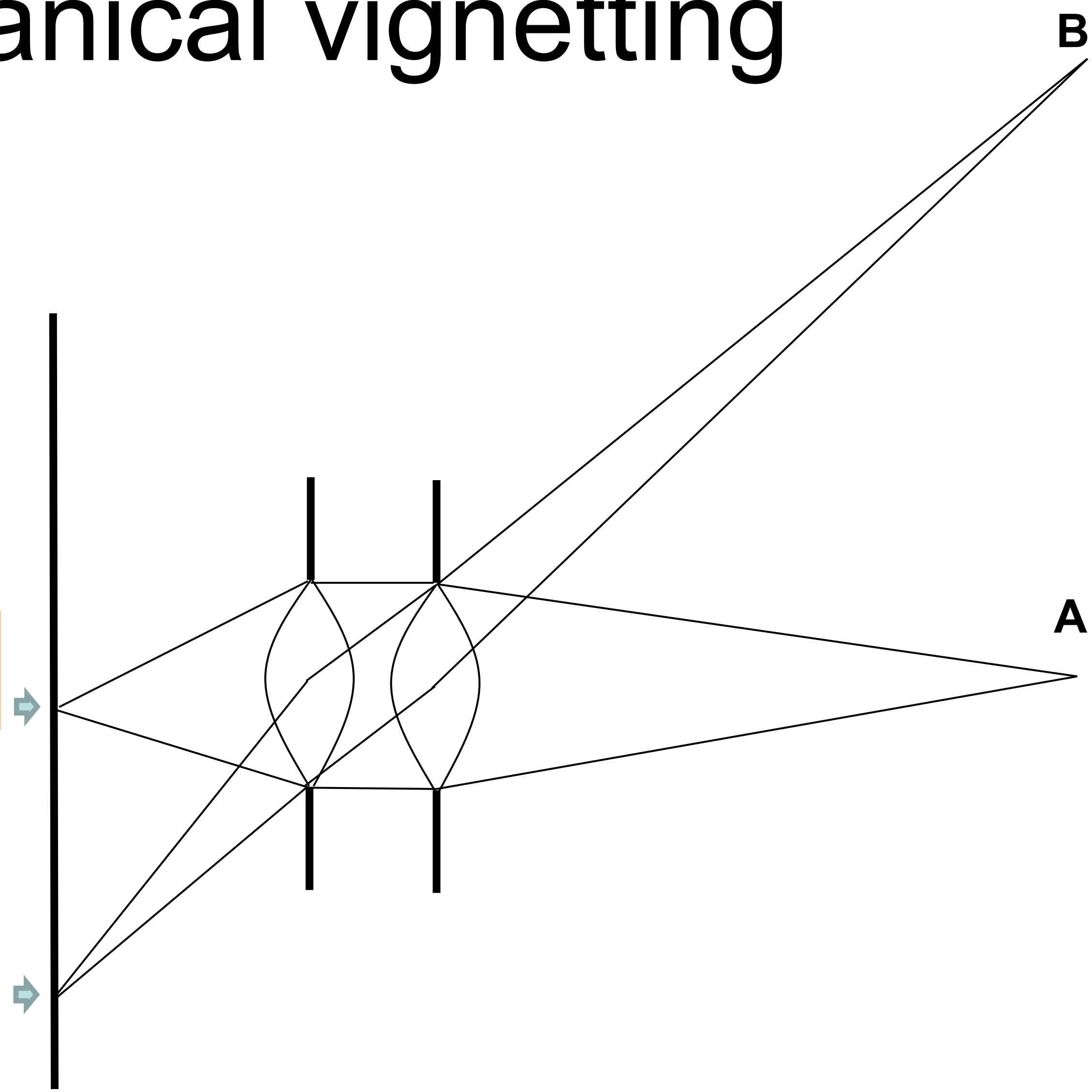
- Even if the light that enters the camera is constant in all directions, the image plane will receive different amount of illumination
- This effect is called *vignetting*

Vignetting

- Sometimes used as a photographic effect
- But is usually unwanted
- Can be compensated for in digital cameras

Image from a digital camera
showing vignetting effect

Mechanical vignetting

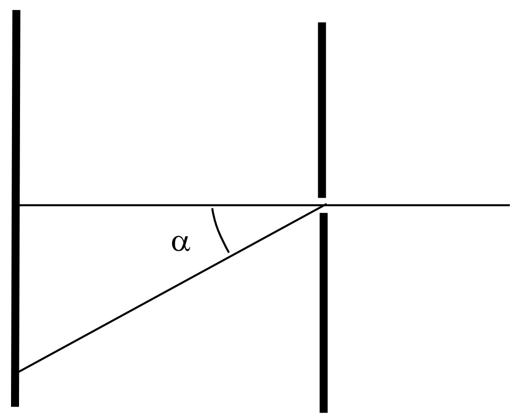


Light from a larger solid angle emitted from point **A** is focused here

Light from a smaller solid angle emitted from point **B** is focused here

The $\cos^4$ law – pinhole camera

- We can see the pinhole as a light source in the form of a small area that illuminates the image plane



  - The flux density decreases with the square of the distance to the light source: $\cos^2 \alpha$
  - The effective area of the detector relative to the pinhole varies as $\cos \alpha$
  - The effective area of the pinhole relative to the detector varies as $\cos \alpha$

The $\cos^4$ law

- This effect exists also in lens-based cameras
- This means that, in general, there is an *attenuation* of the image towards the edges of the image, approximately according to $\cos^4\alpha$
- Can be compensated for in a digital camera

Chromatic aberration

- The refraction index of matter (lenses) is wavelength dependent
  - Example: a prism can decompose the light into its spectrum

  - A ray of white light is decomposed into rays of different colors that intersect the image plane at different points

Chromatic aberration

Sometimes clearly visible if you look at the edges.

End of image formation chapter