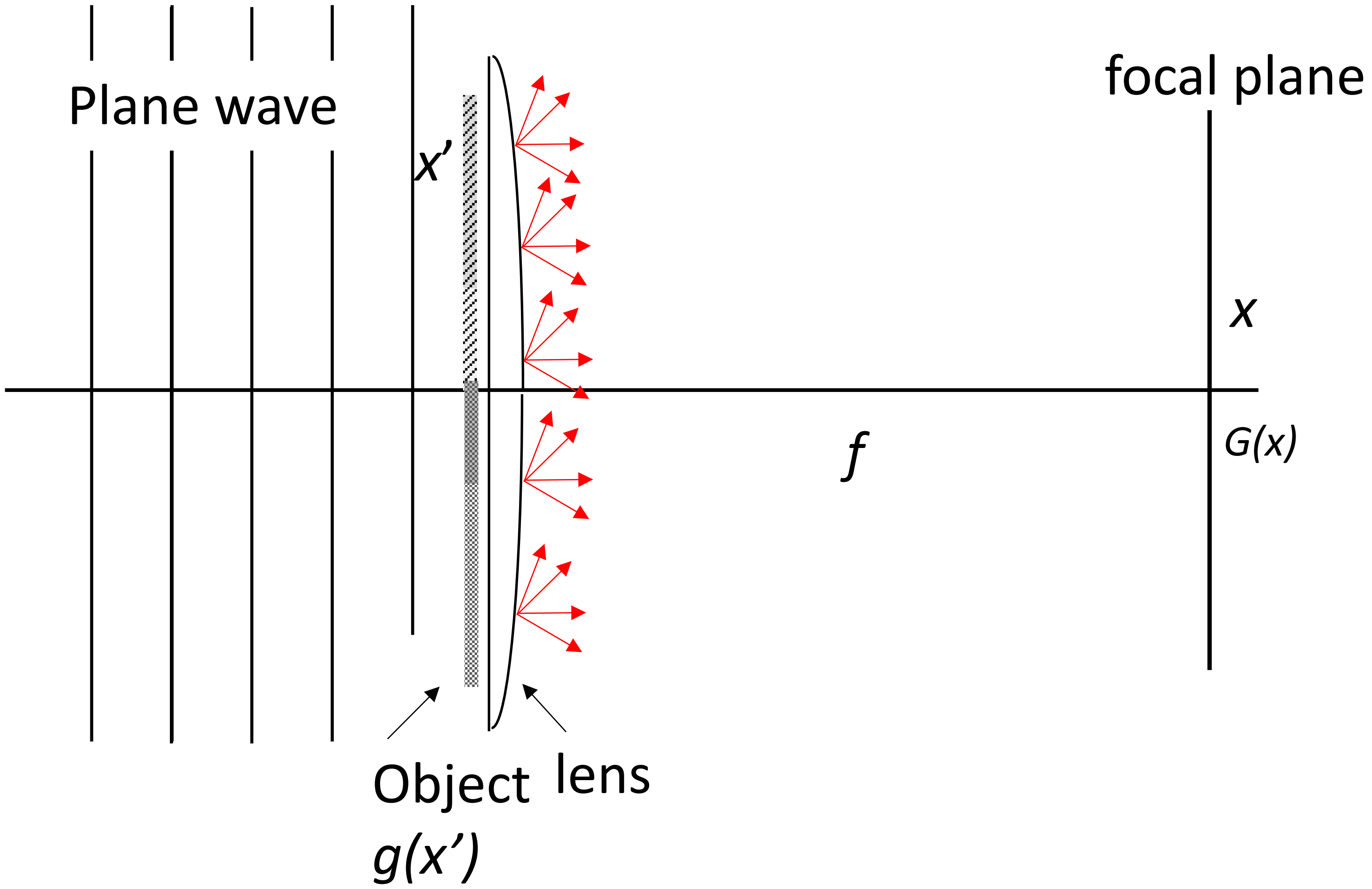
A lens computes the Fourier transform!



- We will show that $G(x)$ is the Fouriertransform of $g(x')$ (apart from a phase factor)
- Using Huygens’ wave model for light.

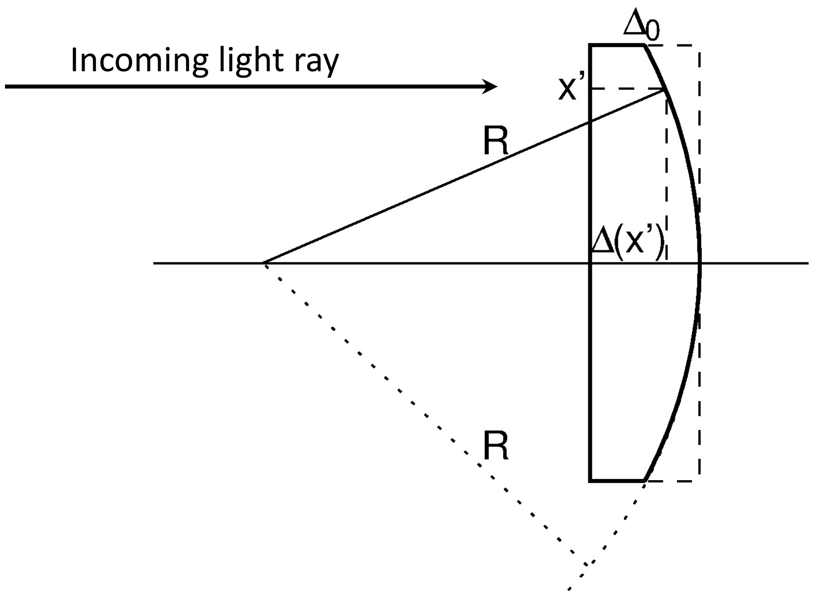
Approach

- Add the light contribution from each point x' entering the lens to each point x in the focal plane, taking magnitude A and phase ϕ into account.
- The contributions consist of light waves of the same wavelength
- Magnitude is given by the object density $g(x')$. Phase depends on the optical path length. Compute this separably for the lens and the path from the lens to the focal plane
- Standard trick: represent each contribution by a complex number $Ae^{i\phi}$
- 1D-analysis (can easily be extended to 2D)

Path length through the lens

Simplifications

- The lens is plano-convex and thin
- Paraxial approximation
- Coherent light
- Inscribe the lens within a virtual rectangular box and apply Huygens' principle on the light coming out from this box.



$$\Delta(x') = \Delta_0 - \left(R - \sqrt{R^2 - x'^2}\right) \approx \Delta_0 - \frac{x'^2}{2R}$$

Where we have used x' << R (paraxial approximation)

Assume that the light travels slower by a factor *n* in glass than in air and exits at the same height (x') since the lens is thin.

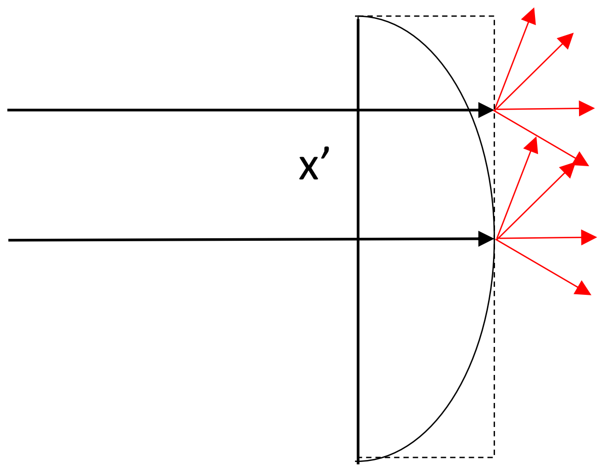
The "optical distance" travelled within the virtual box will then be:

$$\delta(x') = n\Delta(x') + \left(\Delta_0 - \Delta(x')\right) = n\Delta_0 - (n-1)\frac{x'^2}{2R}$$

Applying "Lensmakers Formula" *1/f* = (*n*-1)/*R* gives:

$$\delta(x') = n\Delta_0 - \frac{x'^2}{2f}$$

The phase transform for the lens



The optical pathlength $\delta(x')$ corresponds to the phase shift:

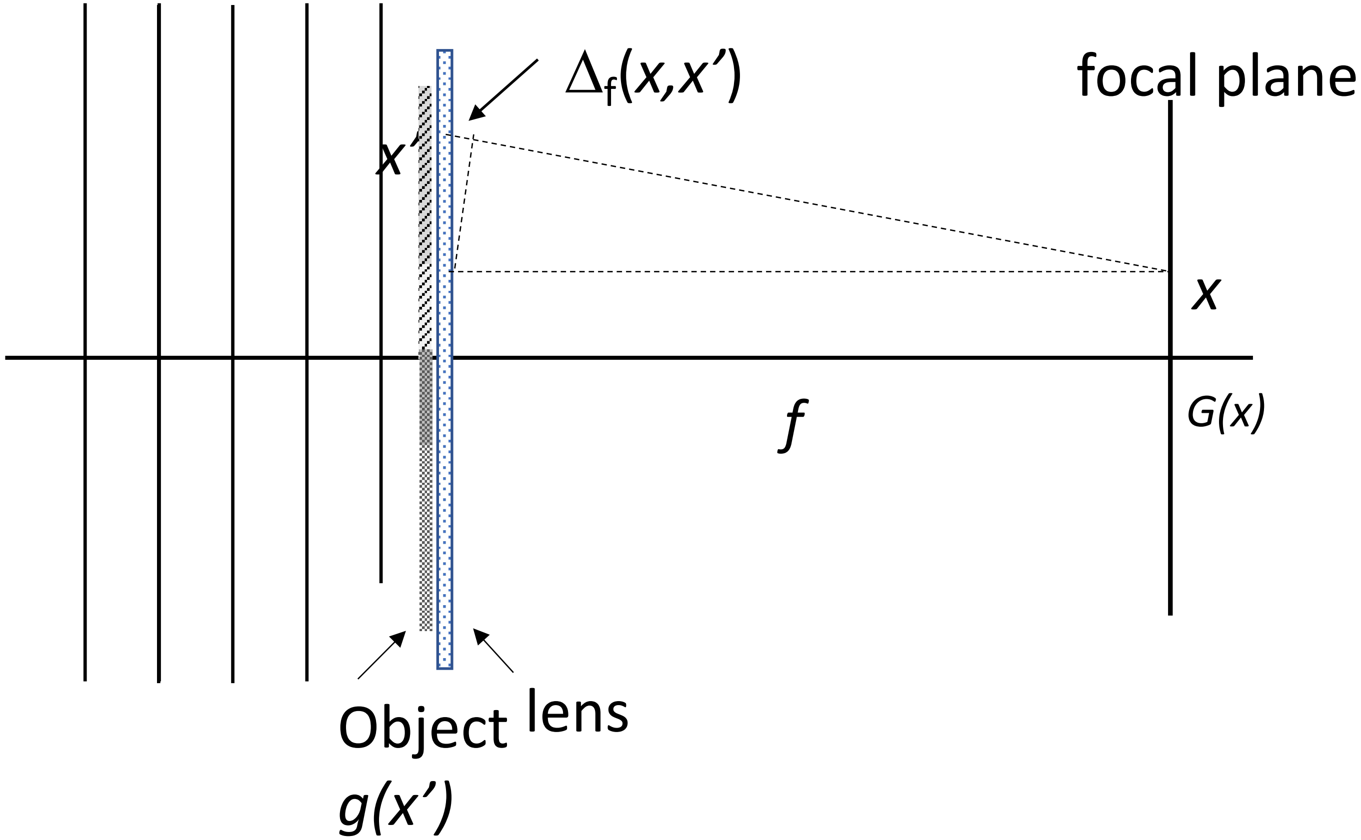
$$\Delta\phi = \frac{2\pi}{\lambda}\delta(x')$$

Inserting $\delta(x') = n\Delta_0 - \frac{x'^2}{2f}$

and disregarding the fixed delay $n\Delta_0$ gives the ”phase transform”:

$$T_L(x') = e^{-i\frac{k}{2f}x'^2} \qquad \text{where} \qquad k = \frac{2\pi}{\lambda}$$

The phase transform from lens to focal plane



$$\Delta_f(x,x') = \sqrt{(x'-x)^2 + f^2} - f \approx \frac{(x'-x)^2}{2f}$$

(again using the paraxial approximation)

Thus $T_f = e^{i\frac{k}{2f}(x'-x)^2}$

Putting it all together

$$G(x) = \int_{-\infty}^{\infty} g(x') \cdot T_L \cdot T_f dx' = \int_{-\infty}^{\infty} g(x') \cdot e^{-i\frac{k}{2f}x'^2} \cdot e^{i\frac{k}{2f}(x'-x)^2} dx' =$$

$$= e^{i\frac{k}{2f}x^2} \int_{-\infty}^{\infty} g(x') e^{-i\frac{k}{f}xx'} dx'$$

Normalise according to *u = x/(λf)*

$$\boldsymbol{G}(u) = e^{i\pi\lambda f u^2} \int_{-\infty}^{\infty} g(x') e^{-i2\pi u x'} dx'$$

This is the Fourier Transform multiplied by a phase factor (of magnitude 1)!

* See the course web how to get rid of the phase factor by putting the object further in front of the lens.